

*selberg and ruelle zeta functions on
compact hyperbolic odd dimensional
manifolds*

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Abstract

In this thesis we study the Selberg and Ruelle zeta functions on compact oriented hyperbolic manifolds X of odd dimension d . These are dynamical zeta functions associated with the geodesic flow on the unit sphere bundle $S(X)$.

Throughout this thesis we identify X with $\Gamma \backslash G/K$, where $G = \mathrm{SO}^0(d, 1)$, $K = \mathrm{SO}(d)$ and Γ is a discrete torsion-free cocompact subgroup of G . Let $G = KAN$ be the Iwasawa decomposition with respect to K . Let M be the centralizer of A in K .

For an irreducible representation σ of M and a finite dimensional representation χ of Γ , we define the Selberg zeta function $Z(s; \sigma, \chi)$ and the Ruelle zeta function $R(s; \sigma, \chi)$. We prove that they converge in some half-plane $\mathrm{Re}(s) > c$ and admit a meromorphic continuation to the whole complex plane. We also describe the singularities of the Selberg zeta function in terms of the discrete spectrum of certain differential operators on X . Furthermore, we provide functional equations relating their values at s with those at $-s$. The main tool that we use is the Selberg trace formula for non-unitary twists. We generalize results of Bunke and Olbrich to the case of non-unitary representations χ of Γ .

γυρισες και μου πες ως τον μαρτη
σ' αλλους παραλληλους θα χεις μπει...

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Introduction

This thesis deals with the dynamical zeta functions of Selberg and Ruelle, defined in terms of the geodesic flow on the unit sphere bundle of a compact oriented hyperbolic manifold of odd dimension. In [GLP13], the Ruelle zeta function has been defined for an Anosov flow on a smooth compact riemannian manifold.

The Ruelle zeta function associated with the geodesic flow on the unit sphere bundle of a closed manifold with C^ω riemannian metric of negative curvature has been studied by Fried in [Fri95]. It is defined by

$$R(s) = \prod_{\gamma} (1 - e^{-sl(\gamma)}),$$

where γ runs over all the prime closed geodesics and $l(\gamma)$ denotes the length of γ . Further In [Fri95, Corollary, p.180], it is proved that it admits a meromorphic continuation to the whole complex plane. In our case, where $X = \Gamma \backslash \mathbb{H}^d$, the dynamical zeta functions are twisted by a representation χ of Γ . They are defined in terms of the lengths of the closed geodesics, also called *length spectrum*.

We begin by giving a short introduction to our algebraic and geometric setting. For all the details, we refer to Chapter 1. For $d \in \mathbb{N}$, $d = 2n + 1$, we let $G = \mathrm{SO}^0(d, 1)$ and $K = \mathrm{SO}(d)$. Let $\tilde{X} = G/K$. \tilde{X} can be equipped with a G -invariant metric, which is unique up to scaling and is of constant negative curvature. If we normalize this metric such that it has constant curvature -1 , then \tilde{X} , equipped with this metric, is isometric to \mathbb{H}^d . Let $\Gamma \subset G$ be a discrete torsion-free subgroup such that $\Gamma \backslash G$ is compact. Then Γ acts by isometries on \tilde{X} and $X = \Gamma \backslash \tilde{X}$ is a compact oriented hyperbolic manifold of dimension d . Note that G has real rank 1. This means that in the Iwasawa decomposition $G = KAN$, A is a multiplicative torus of dimension 1, i.e., $A \cong \mathbb{R}^+$.

The Ruelle and Selberg zeta functions are defined as follows. For a given $\gamma \in \Gamma$ we denote by $[\gamma]$ the Γ -conjugacy class of γ . The conjugacy class $[\gamma]$ is called prime if there exist no $k > 1$ and $\gamma_0 \in \Gamma$ such that $\gamma = \gamma_0^k$. If $\gamma \neq e$, then there is a unique closed geodesic c_γ associated with $[\gamma]$. Let $l(\gamma)$ denote the length of c_γ . We associate to every prime conjugacy class $[\gamma]$ the so called *prime geodesic*. Let M be the centralizer of A in K . Let also \mathfrak{g} , \mathfrak{n} and \mathfrak{a} be the Lie algebras of G , N

and A correspondingly. Let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be the Cartan decomposition of \mathfrak{g} . There is an isomorphism $\mathfrak{p} \cong T_{eK}\tilde{X}$. We denote by \widehat{M} the set of equivalence classes of irreducible unitary representations of M . Let $H \in \mathfrak{a}$ be of norm 1 and positive with respect to the choice of N . Then, for every $\gamma \in \Gamma - \{e\}$ there exist $g \in G$, $a_\gamma = \exp l(\gamma)H \in A$, and $m_\gamma \in M$ such that $g\gamma g^{-1} = m_\gamma a_\gamma$, where a_γ depends only on γ and m_γ is unique up to conjugation in M ([Wal76, Lemma 6.6]). We define the zeta functions depending on representations of M and Γ .

Definition A. Let $\chi: \Gamma \rightarrow \mathrm{GL}(V_\chi)$ be a finite dimensional representation of Γ . Let $\sigma \in \widehat{M}$. Then, the twisted Selberg zeta function $Z(s; \sigma, \chi)$ is defined by the infinite product

$$Z(s; \sigma, \chi) := \prod_{\substack{[\gamma] \neq e, \\ [\gamma] \text{ prime}}} \prod_{k=0}^{\infty} \det \left(\mathrm{Id} - (\chi(\gamma) \otimes \sigma(m_\gamma) \otimes S^k(\mathrm{Ad}(m_\gamma a_\gamma)_{\bar{\mathfrak{n}}})) e^{-(s+|\rho|)l(\gamma)} \right),$$

where $s \in \mathbb{C}$, $\bar{\mathfrak{n}} = \theta \mathfrak{n}$ is the sum of the negative root spaces of \mathfrak{a} and $S^k(\mathrm{Ad}(m_\gamma a_\gamma)_{\bar{\mathfrak{n}}})$ denotes the k -th symmetric power of the adjoint map $\mathrm{Ad}(m_\gamma a_\gamma)$ restricted to $\bar{\mathfrak{n}}$.

Definition B. Let $\chi: \Gamma \rightarrow \mathrm{GL}(V_\chi)$ be a finite dimensional representation of Γ . Let $\sigma \in \widehat{M}$. Then, the twisted Ruelle zeta function $R(s; \sigma, \chi)$ is defined by the infinite product

$$R(s; \sigma, \chi) := \prod_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \det(\mathrm{Id} - \chi(\gamma) \otimes \sigma(m_\gamma) e^{-sl(\gamma)})^{(-1)^{d-1}}.$$

For unitary representations χ of Γ , these zeta functions have been studied by Fried ([Fri86]) and Bunke and Olbrich ([BO95]). However, for the applications (cf. [Mül12b]), it is important to have results available for general finite dimensional representations.

In Fried ([Fri86]) the zeta functions have been studied explicitly for a closed oriented hyperbolic manifold X of dimension d . He considers the standard representation of $M = \mathrm{SO}(d-1)$ on $\Lambda^j \mathbb{C}^{d-1}$ and an orthogonal representation $\rho: \Gamma \rightarrow \mathrm{O}(\mathfrak{m})$ of Γ . Using the Selberg trace formula for the heat operator $e^{-t\Delta_j}$, where Δ_j is the Hodge Laplacian on j -forms on X , he managed to prove the meromorphic continuation of the zeta functions to the whole complex plane \mathbb{C} , as well as functional equations for the Selberg zeta function ([Fri86, p.531-532]). He proved also the following theorem, in the case of $d = \dim(X)$ being odd and ρ acyclic, i.e. the twisted cohomology groups $H^*(X; \rho)$ vanish for all j .

Theorem ([Fri86, Theorem 1]). *Let $X = \Gamma \backslash \mathbb{H}^d$ be a compact oriented hyperbolic manifold of odd dimension. Assume that $\rho: \Gamma \rightarrow \mathrm{O}(\mathfrak{m})$ is acyclic. Then, the*

Ruelle zeta function

$$R(s; \rho) = \prod_{\substack{[\gamma] \neq e, \\ [\gamma] \text{ prime}}} \det(\text{Id} - \rho(\gamma)e^{-s l(\gamma)}),$$

which converges for $\text{Re}(s) > d - 1$, admits a meromorphic extension to \mathbb{C} . It is holomorphic at $s = 0$ and for $\varepsilon = (-1)^{d-1}$

$$|R(0; \rho)^\varepsilon| = T_X(\rho)^2,$$

where $T_X(\rho)$ is the Ray-Singer analytic torsion defined in [RS71].

This theorem is of interest, since it connects the Ruelle zeta function evaluated at zero with the analytic torsion under certain assumptions.

Question 1. *How can one generalize these results for a non-unitary representation of Γ in the case of a compact hyperbolic odd dimensional manifold?*

Wotzke dealt with this conjecture in his thesis ([Wot08]). More specifically, he considered a finite dimensional complex representation $\tau : G \rightarrow \text{GL}(V)$ of G and its restrictions $\tau|_K$ and $\tau|_\Gamma$ to K and Γ , respectively. By [MM63, Proposition 3.1], there exists an isomorphism between the locally homogenous vector bundle E_τ over X associated with $\tau|_K$ and the flat vector bundle E_{fl} over X associated with $\tau|_\Gamma$, i.e.

$$\Gamma \backslash (G/K \times V) \cong (\Gamma \backslash G \times V)/K. \quad (1)$$

Then, by [MM63, Lemma 3.1], there exists a hermitian inner product on V , which is unique up to scaling and, in particular, is skew-symmetric with respect to \mathfrak{k} . Hence, it defines a fiber metric in E_τ , which by (1) descends to a fiber metric in E_{fl} . He considered the Hodge-Laplace operator $\Delta_r(\tau)$ acting on r -forms on X with values in E_{fl} . Using again the isomorphism (1), he considered the Hodge-Laplace operator acting on $(C^\infty(\Gamma \backslash G) \otimes \Lambda^p \mathfrak{p}^* \otimes V)^K$. He proved the meromorphic continuation of the Selberg zeta function using the Selberg trace formula for the operator $e^{-t\Delta_r(\tau)}$ (specifically he considered the function $\sum_{r=0}^d (-1)^r r \text{Tr}(e^{-t\Delta_r(\tau)})$) and the connection of the logarithmic derivative of the Selberg zeta function to the hyperbolic contribution in the trace formula. As a generalization of the equation (RS) in Fried ([Fri86, p.532]), he proved a product formula, which expresses the Ruelle zeta function as product of Selberg zeta functions with shifted origins:

$$R(s; \tau|_\Gamma) = \prod_{w \in W^1} Z(s + \lambda_\tau(w); \nu_\tau(w))^{(-1)^{l(w)+1}}, \quad (2)$$

where W^1 is a subgroup of the Weyl group W_G , $\lambda_\tau(w)$ is a number defined by the action of the Weyl group W^1 on the highest weight of τ and $\nu_\tau(w)$ is an irreducible

representation of M associated with τ (cf. [Wot08, p.40]). Hence, by (2), Wotzke obtained the meromorphic continuation of the Ruelle zeta function. Further, as a generalization of equation (14) in Fried ([Fri86, p.535]), he proved a determinant formula that connects the Selberg zeta function and the regularized determinant of certain Laplace-type operators $\Delta(w)$ associated to the representation $\nu_\tau(w)$:

$$S(s; w) = \det_s(\Delta(w) - \lambda_\tau(w)^2 + s^2) \exp\left(-2\pi \text{Vol}(X) \int_0^s P(\lambda; w) d\lambda\right),$$

where $S(z; w)$ denotes the symmetrized zeta function (cf. equation (4)) and $P(\lambda; w)$ denotes the Plancherel polynomial. With the additional assumption that $\tau \neq \tau_\theta$, where $\tau_\theta = \tau \circ \theta$ and θ denotes the Cartan involution of G , the following theorem was proved.

Theorem ([Wot08, Theorem 8.13]). *Let $\tau \neq \tau_\theta$. Then the Ruelle zeta function $R(s; \tau|_\Gamma)$ is regular at $s = 0$ and*

$$|R(0; \tau|_\Gamma)| = T_X(\tau|_\Gamma)^2.$$

Question 2. *How can one generalize these results for an arbitrary non-unitary representation of Γ ?*

In our case, we consider an arbitrary finite dimensional representation $\chi: \Gamma \rightarrow \text{GL}(V_\chi)$ of Γ . Our approach to the problem of proving the meromorphic continuation and functional equations for both the Selberg and Ruelle zeta functions is different from the method of Wotzke, since we consider an arbitrary representation of Γ and can not apply the isomorphism (1).

Our results can be viewed as a generalization of the results in the book of Bunke and Olbrich ([BO95]). Again, since we consider a non-unitary representation of Γ we have to deal with several problems and consider additional theory to solve them.

First, the convergence of the zeta functions in some half plane is not trivial. We use the word metric on Γ to prove the following propositions.

Proposition C. *Let $\chi: \Gamma \rightarrow \text{GL}(V_\chi)$ be a finite dimensional representation of Γ . Then, there exists a constant $c > 0$ such that*

$$Z(s; \sigma, \chi) := \prod_{\substack{[\gamma] \neq e, \\ [\gamma] \text{ prime}}} \prod_{k=0}^{\infty} \det(\text{Id} - (\chi(\gamma) \otimes \sigma(m_\gamma) \otimes S^k(\text{Ad}(m_\gamma a_\gamma)_{\bar{\mathbb{N}}})) e^{-(s+1)l(\gamma)})$$

converges absolutely and uniformly on compact subsets of the half-plane $\text{Re}(s) > c$.

Proposition D. *Let $\chi : \Gamma \rightarrow \mathrm{GL}(V_\chi)$ be a finite dimensional representation of Γ . Then, there exists a constant $c > 0$ such that*

$$R(s; \sigma, \chi) := \prod_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \det(\mathrm{Id} - \chi(\gamma) \otimes \sigma(m_\gamma) e^{-sl(\gamma)})^{(-1)^{d-1}}.$$

converges absolutely and uniformly on compact subsets of the half-plane $\mathrm{Re}(s) > c$.

Secondly, if we consider an arbitrary representation χ of Γ , there is no hermitian metric on the associated flat vector bundle $E_\chi = \tilde{X} \times_X V_\chi \rightarrow X$ which is compatible with the flat connection. In order to overcome this problem we use the concept of the flat Laplacian (cf. Chapter 4, Sections 4.1, and 4.2). This operator was first introduced by Müller in [Mül11]. We give here a short description of this operator.

Let $\tau : K \rightarrow \mathrm{GL}(V_\tau)$ be a complex finite dimensional unitary representation of K . Let $\tilde{E}_\tau := G \times_\tau V_\tau \rightarrow \tilde{X}$ be the associated homogenous vector bundle over \tilde{X} . Let $E_\tau := \Gamma \backslash (G \times_\tau V_\tau) \rightarrow X$ be the locally homogenous vector bundle over X . Let Δ_τ be the Bochner-Laplace operator associated with the canonical connection on E_τ (cf. Chapter 4, Section 4.2). We define the operator $\Delta_{\tau, \chi}^\sharp$ acting on $C^\infty(X, E_\tau \otimes E_\chi)$ as follows. Locally, with respect to any basis of flat sections, the operator takes the form

$$\tilde{\Delta}_{\tau, \chi}^\sharp = \tilde{\Delta}_\tau \otimes \mathrm{Id}_{V_\chi}, \quad (3)$$

where $\tilde{\Delta}_{\tau, \chi}^\sharp$ and $\tilde{\Delta}_\tau$ are the lifts to \tilde{X} of $\Delta_{\tau, \chi}^\sharp$ and Δ_τ , respectively.

Contrary to the settings of Wotzke and Bunke and Olbrich, our operator is not self-adjoint. However, it still has nice spectral properties, i.e., the spectrum of $\Delta_{\tau, \chi}^\sharp$ is a discrete subset of a positive cone in \mathbb{C} (as in Figure A.1, Appendix A). We consider the corresponding heat semi-group $e^{-t\Delta_{\tau, \chi}^\sharp}$ acting on the space of smooth sections of the vector bundle $E_\tau \otimes E_\chi$. It is an integral operator with smooth kernel $H_t^{\tau, \chi} \in C^\infty(X, (E_\tau \otimes E_\chi) \otimes (E_\tau \otimes E_\chi)^*)$, which can be expressed as

$$H_t^{\tau, \chi}(x, y) = \sum_{\gamma \in \Gamma} H_t^\tau(\tilde{x}, \gamma \tilde{y}) \otimes \chi(\gamma) \mathrm{Id}_{V_\chi},$$

where \tilde{x}, \tilde{y} are lifts of x, y to \tilde{X} , respectively, and H_t^τ is the kernel of $e^{-t\tilde{\Delta}_\tau}$. Note that H_t^τ belongs to the space $C^\infty(\tilde{X} \times \tilde{X}, \tilde{E}_\tau \boxtimes \tilde{E}_\tau^*)$. Since $H_t^\tau(\tilde{x}, \tilde{y})$ is G -invariant, it corresponds to a convolution operator, given by a kernel

$$\tilde{H}_t^\tau : G \rightarrow \mathrm{End}(V_\tau).$$

This kernel belongs to the space $(\mathcal{C}^q(G) \otimes \mathrm{End}(V_\tau))^{K \times K}$, for $t > 0$, where $\mathcal{C}^q(G)$ denotes the Harish-Chandra L^q -Schwartz space for every $q > 0$ (cf. Section 3.2 for

the definition of this space).

Hence, we can consider the trace of the operator $e^{-t\Delta_{\tau,x}^\sharp}$ and derive a corresponding trace formula. By [Mül11, Proposition 4.1], we have the following proposition.

Proposition E (Selberg trace formula for non-unitary representations). *Let E_χ be a flat vector bundle over $X = \Gamma \backslash \tilde{X}$, associated with a finite dimensional complex representation $\chi: \Gamma \rightarrow \mathrm{GL}(V_\chi)$ of Γ . Let $\Delta_{\tau,\chi}^\sharp$ be the twisted Bochner-Laplace operator acting on $C^\infty(X, E_\tau \otimes E_\chi)$. Then,*

$$\mathrm{Tr}(e^{-t\Delta_{\tau,\chi}^\sharp}) = \sum_{\gamma \in \Gamma} \mathrm{tr} \chi(\gamma) \int_{\Gamma \backslash G} \mathrm{tr} \tilde{H}_t^\tau(g^{-1}\gamma g) dg.$$

In fact, we use specific twisted Bochner-Laplace-type operators $A_\chi^\sharp(\sigma)$, induced by $\Delta_{\tau,\chi}^\sharp$. These operators are defined as follows. We already associated the Selberg and Ruelle zeta functions with irreducible representations σ of M . These representations are chosen precisely to be the representations arising from restrictions of representations of K . Let $i^*: R(K) \rightarrow R(M)$ be the pullback of the embedding $i: M \hookrightarrow K$, where $R(K)$, $R(M)$ denote the representation rings over \mathbb{Z} of K and M , respectively. Throughout this thesis, we will distinguish the following two cases:

- **case (a)**: σ is invariant under the action of the restricted Weyl group W_A .
- **case (b)**: σ is not invariant under the action of the restricted Weyl group W_A .

The trace formulas, the results concerning the meromorphic continuation of the zeta functions and the functional equations will be derived under this distinction. In case (b), we consider the symmetrized zeta function

$$S(s; \sigma, \chi) := Z(s; \sigma, \chi)Z(ws; \sigma, \chi), \quad (4)$$

the super Selberg zeta function

$$Z^s(s; \sigma, \chi) := \frac{Z(s; \sigma, \chi)}{Z(s; w\sigma, \chi)},$$

and the super Ruelle zeta function

$$R^s(s; \sigma, \chi) := \frac{R(s; \sigma, \chi)}{R(s; w\sigma, \chi)},$$

where w is a non-trivial element of the restricted Weyl group W_A .

In both cases we construct a graded vector bundle $E(\sigma)$ over X in the following way. By [BO95, Proposition 1.1], we know that there exist unique integers $m_\tau(\sigma) \in \{-1, 0, 1\}$, which are equal to zero except for finitely many $\tau \in \hat{K}$, such that for

- case (a)

$$\sigma = \sum_{\tau \in \widehat{K}} m_\tau(\sigma) i^*(\tau); \quad (5)$$

- case (b)

$$\sigma + w\sigma = \sum_{\tau \in \widehat{K}} m_\tau(\sigma) i^*(\tau). \quad (6)$$

Then, the locally homogeneous vector bundle $E(\sigma)$ associated with τ is defined as

$$E(\sigma) = \bigoplus_{\substack{\tau \in \widehat{K} \\ m_\tau(\sigma) \neq 0}} E_\tau, \quad (7)$$

where E_τ is the locally homogeneous vector bundle associated with $\tau \in \widehat{K}$. Using the sign of $m_\tau(\sigma)$, we obtain a grading $E(\sigma) = E(\sigma)^+ \oplus E(\sigma)^-$ on the vector bundle $E(\sigma)$ (cf. Section 4.3 for further details).

We consider the operator $A_\tau := -R(\Omega)$ on $C^\infty(X, E_\tau)$, induced by the Casimir element Ω . We define the operator $A_{\tau, \chi}^\sharp$ in a similar way as the twisted Bochner-Laplace operator $\Delta_{\tau, \chi}^\sharp$ in (3). Namely,

$$\widetilde{A}_{\tau, \chi}^\sharp = \widetilde{A}_\tau \otimes \text{Id}_{V_\chi},$$

where $\widetilde{A}_{\tau, \chi}^\sharp, \widetilde{A}_\tau$ denote the lifts to \widetilde{X} of $A_{\tau, \chi}^\sharp, A_\tau$, respectively. We define the operator $A_\chi^\sharp(\sigma)$ acting on smooth sections of $E(\sigma) \otimes E_\chi$ by

$$A_\chi^\sharp(\sigma) := \bigoplus_{m_\tau(\sigma) \neq 0} A_{\tau, \chi}^\sharp + c(\sigma),$$

where $c(\sigma)$ is a number defined by the highest weight of σ .

Theorem F (trace formula for the operator $e^{-tA_\chi^\sharp(\sigma)}$). *For every $\sigma \in \widehat{M}$ we have*

- case (a)

$$\begin{aligned} \text{Tr}(e^{-tA_\chi^\sharp(\sigma)}) &= \dim(V_\chi) \text{Vol}(X) \int_{\mathbb{R}} e^{-t\lambda^2} P_\sigma(i\lambda) d\lambda \\ &+ \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} L_{\text{sym}}(\gamma; \sigma) \frac{e^{-l(\gamma)^2/4t}}{(4\pi t)^{1/2}}, \end{aligned}$$

- **case(b)**

$$\begin{aligned} \mathrm{Tr}(e^{-tA_\chi^\sharp(\sigma)}) &= 2 \dim(V_\chi) \mathrm{Vol}(X) \int_{\mathbb{R}} e^{-t\lambda^2} P_\sigma(i\lambda) d\lambda \\ &\quad + \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} L_{\mathrm{sym}}(\gamma; \sigma + w\sigma) \frac{e^{-l(\gamma)^2/4t}}{(4\pi t)^{1/2}}, \end{aligned}$$

where

$$L_{\mathrm{sym}}(\gamma; \sigma) = \frac{\mathrm{tr}(\sigma(m_\gamma) \otimes \chi(\gamma)) e^{-|\rho|l(\gamma)}}{\det(\mathrm{Id} - \mathrm{Ad}(m_\gamma a_\gamma)_{\bar{\mathfrak{n}}})}.$$

Next, we define the twisted Dirac operator $D_\chi^\sharp(\sigma)$ acting on $C^\infty(X, E_{\tau_s(\sigma)} \otimes E_\chi)$. We let $K = \mathrm{Spin}(d)$, s be the spin representation of K and $\tau(\sigma) \in \widehat{K}$ (cf. Section 5.1). $E_{\tau_s(\sigma)}$ denotes the locally homogenous vector bundle over X , associated with $\tau_s(\sigma) := s \otimes \tau(\sigma)$. Locally, with respect to any basis of flat sections, the operator takes the form

$$\widetilde{D}_\chi^\sharp(\sigma) = \widetilde{D}(\sigma) \otimes \mathrm{Id}_{V_\chi},$$

where $\widetilde{D}_\chi^\sharp(\sigma), \widetilde{D}(\sigma)$ are the lifts to \widetilde{X} of $D_\chi^\sharp(\sigma), D(\sigma)$, respectively, and $D(\sigma)$ is the Dirac operator associated with the representation $\tau_s(\sigma)$ of K . We consider the trace class operator $D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}$ and derive a corresponding trace-formula.

Theorem G (trace formula for the operator $D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}$). *For every $\sigma \in \widehat{M}$ we have*

$$\mathrm{Tr}(D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) = \sum_{[\gamma] \neq e} \frac{-2\pi i}{(4\pi t)^{3/2}} \frac{l^2(\gamma) \mathrm{tr}(\chi(\gamma) \otimes (\sigma(m_\gamma) - w\sigma(m_\gamma)))}{n_\Gamma(\gamma) D(\gamma)} e^{-l^2(\gamma)/4t}.$$

We define the operators $R(s_i^2) := ((\square) + s_i^2)^{-1}$, $s_i \in \mathbb{C} - \mathrm{spec}(\square)$, where $\square = A_\chi^\sharp(\sigma)$ or $D_\chi^\sharp(\sigma)$. Let $N \in \mathbb{N}$. We use the generalized resolvent identity (Lemma 6.1, Section 6.1):

$$\prod_{i=1}^N R(s_i^2) = \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) R(s_i^2),$$

The trace formulas in Theorem F and Theorem G together with this identity will be the main tools to prove our results. The proofs of the meromorphic continuation of the zeta functions are based on the fact that if we insert the right hand side of the trace formulas for the operators $P_t = e^{-tA_\chi^\sharp(\sigma)}$ or $D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}$ in the integral

$$\int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \mathrm{Tr} P_t dt,$$

then the integral that includes the term of the hyperbolic contribution, arising from the trace formula for P_t , is related to

- **case (a)** $\frac{d}{ds} \log Z(s; \sigma, \chi)$, if $P_t = e^{-tA_\chi^\sharp(\sigma)}$.
- **case (b)** $\frac{d}{ds} \log S(s; \sigma, \chi)$, if $P_t = e^{-tA_\chi^\sharp(\sigma)}$ and $\frac{d}{ds} \log Z^s(s; \sigma, \chi)$, if $P_t = D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}$.

We state our main results. In Theorems H and I, we choose the branch of the square roots of the complex numbers t_k and μ_k , respectively, whose real part is positive. In addition, if t_k and μ_k are negative real numbers, we choose the branch of the square roots, whose imaginary part is positive.

Meromorphic continuation of the Selberg zeta function

- **case (a)**

Theorem H. *The Selberg zeta function $Z(s; \sigma, \chi)$ admits a meromorphic continuation to the whole complex plane \mathbb{C} . The set of the singularities equals $\{s_k^\pm = \pm i\sqrt{t_k} : t_k \in \text{spec}(A_\chi^\sharp(\sigma)), k \in \mathbb{N}\}$. The orders of the singularities are equal to $m(t_k)$, where $m(t_k) \in \mathbb{N}$ denotes the algebraic multiplicity of the eigenvalue t_k . For $t_0 = 0$, the order of the singularity s_0 is equal to $2m(0)$.*

- **case (b)**

Theorem I. *The symmetrized zeta function $S(s; \sigma, \chi)$ admits a meromorphic continuation to the whole complex plane \mathbb{C} . The set of the singularities equals $\{s_k^\pm = \pm i\sqrt{\mu_k} : \mu_k \in \text{spec}(A_\chi^\sharp(\sigma)), k \in \mathbb{N}\}$. The orders of the singularities are equal to $m(\mu_k)$, where $m(\mu_k) \in \mathbb{N}$ denotes the algebraic multiplicity of the eigenvalue μ_k . For $\mu_0 = 0$, the order of the singularity s_0 is equal to $2m(0)$.*

Theorem J. *The super zeta function $Z^s(s; \sigma, \chi)$ admits a meromorphic continuation to the whole complex plane \mathbb{C} . The singularities are located at $\{s_k^\pm = \pm i\lambda_k : \lambda_k \in \text{spec}(D_\chi^\sharp(\sigma)), k \in \mathbb{N}\}$ of order $\pm m_s(\lambda_k)$, where $m_s(\lambda_k) = m(\lambda_k) - m(-\lambda_k) \in \mathbb{N}$ and $m(\lambda_k)$ denotes the algebraic multiplicity of the eigenvalue λ_k .*

Theorem K. *The Selberg zeta function $Z(s; \sigma, \chi)$ admits a meromorphic continuation to the whole complex plane \mathbb{C} . The set of the singularities equals to $\{s_k^\pm = \pm i\lambda_k : \lambda_k \in \text{spec}(D_\chi^\sharp(\sigma)), k \in \mathbb{N}\}$. The orders of the singularities are equal to $\frac{1}{2}(\pm m_s(\lambda_k) + m(\lambda_k^2))$. For $\lambda_0 = 0$, the order of the singularity is equal to $m(0)$.*

Meromorphic continuation of the Ruelle zeta function

Theorem L. *For every $\sigma \in \widehat{M}$, the Ruelle zeta function $R(s; \sigma, \chi)$ admits a meromorphic continuation to the whole complex plane \mathbb{C} .*

Functional equations

- **case (a)**

Theorem M. *The Selberg zeta function $Z(s; \sigma, \chi)$ satisfies the functional equation*

$$\frac{Z(s; \sigma, \chi)}{Z(-s; \sigma, \chi)} = \exp \left(-4\pi \dim(V_\chi) \text{Vol}(X) \int_0^s P_\sigma(r) dr \right),$$

where P_σ denotes the Plancherel polynomial associated with $\sigma \in \widehat{M}$.

- **case (b)**

Theorem N. *The symmetrized zeta function $S(s; \sigma, \chi)$ satisfies the functional equation*

$$\frac{S(s; \sigma, \chi)}{S(-s; \sigma, \chi)} = \exp \left(-8\pi \dim(V_\chi) \text{Vol}(X) \int_0^s P_\sigma(r) dr \right),$$

where P_σ denotes the Plancherel polynomial associated with $\sigma \in \widehat{M}$.

Theorem O. *The super zeta function $Z^s(s, \sigma, \chi)$ satisfies the functional equation*

$$Z^s(s; \sigma, \chi) Z^s(-s; \sigma, \chi) = e^{2\pi i \eta(0, D_\chi^\sharp(\sigma))},$$

where $\eta(0, D_\chi^\sharp(\sigma))$ denotes the eta invariant associated with the Dirac operator $D_\chi^\sharp(\sigma)$. Furthermore,

$$Z^s(0; \sigma, \chi) = e^{\pi i \eta(0, D_\chi^\sharp(\sigma))}.$$

Theorem P. *The Ruelle zeta function satisfies the functional equation*

$$\frac{R(s; \sigma, \chi)}{R(-s; \sigma, \chi)} = \exp \left(-4\pi(d+1) \dim(V_\sigma) \dim(V_\chi) \text{Vol}(X)s \right). \quad (8)$$

Theorem Q. *The super Ruelle zeta function, associated with a non-Weyl invariant representation $\sigma \in \widehat{M}$, satisfies the functional equation*

$$R^s(s; \sigma, \chi) R^s(-s; \sigma, \chi) = e^{2i\pi\eta(D_\chi^\sharp(\sigma \otimes \sigma_p))}, \quad (9)$$

where σ_p denotes the p -th exterior power of the standard representation of M , and $\eta(D_\chi^\sharp(\sigma \otimes \sigma_p))$ the eta invariant of the twisted Dirac-operator $D_\chi^\sharp(\sigma \otimes \sigma_p)$.

Moreover, the following equation holds:

$$\frac{R(s; \sigma, \chi)}{R(-s; w\sigma, \chi)} = e^{i\pi\eta(D_\chi^\sharp(\sigma \otimes \sigma_p))} \exp\left(-4\pi(d+1)\dim(V_\sigma)\dim(V_\chi)\text{Vol}(X)s\right). \quad (10)$$

This thesis is organized as follows. In Chapter 1, we provide the basic set up, which is needed, concerning the compact hyperbolic odd dimensional manifolds.

In Chapter 2, we introduce the twisted Ruelle and Selberg zeta functions associated with an arbitrary finite dimensional representation χ of Γ and $\sigma \in \widehat{M}$. We prove the convergence of the zeta functions on some half-plane of \mathbb{C} .

Chapter 3 describes the trace formula for integral operators for all locally symmetric spaces of real rank 1. The trace formula which we will derive is

$$\begin{aligned} \text{Tr } R_\Gamma(h) &= \dim(V_\chi) \text{Vol}(X)h(e) + \\ &\sum_{[\gamma] \neq e} \sum_{\sigma \in \widehat{M}} \frac{\overline{\text{tr}(\sigma)} \text{tr}(\chi(\gamma))l(\gamma_0)}{2\pi D(m_\gamma a_\gamma)} \int_{\mathbb{R}} \Theta_{\sigma, \lambda}(h) e^{-i\lambda l(\gamma)} d\lambda. \end{aligned} \quad (11)$$

In Chapter 4, we study the twisted Bochner-Laplace operator $\Delta_{\tau, \chi}^\sharp$ associated to a complex finite dimensional unitary representation τ of K and a complex finite dimensional non-unitary representation of Γ . We define the operator $A_\chi^\sharp(\sigma)$ induced by $\Delta_{\tau, \chi}^\sharp(\sigma)$. In the proof of Theorem F, we use formula (11) but now χ is a non-unitary representation of Γ .

Chapter 5 deals with the twisted Dirac operator $D_\chi^\sharp(\sigma)$ associated to a representation $\tau_s(\sigma) \in \widehat{K}$ and an arbitrary representation χ of Γ . We derive the corresponding trace formula for the operator $D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2}$. Furthermore, we define the eta function $\eta(s, D_\chi^\sharp(\sigma))$ of the operator $D_\chi^\sharp(\sigma)$, and we prove the following equation

$$\eta(s, D_\chi^\sharp(\sigma)) = \eta_0(s, D_\chi^\sharp(\sigma)) + \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty \text{Tr}(\Pi_+ D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) t^{\frac{s-1}{2}} dt, \quad (12)$$

where Π_+ is the projection on the span of the root spaces corresponding to eigenvalues λ with $\text{Re}(\lambda^2) > 0$, and $\eta_0(s, D_\chi^\sharp(\sigma))$ is defined by

$$\eta_0(s, D_\chi^\sharp(\sigma)) := \sum_{\substack{\text{Re}(\lambda) > 0 \\ \text{Re}(\lambda^2) \leq 0}} \lambda^{-s} - \sum_{\substack{\text{Re}(\lambda) < 0 \\ \text{Re}(\lambda^2) \leq 0}} \lambda^{-s},$$

(cf. Lemma 5.5). This relation is not a trivial fact, since the twisted Dirac operator $D_\chi^\sharp(\sigma)$ is not a self-adjoint operator, and therefore its spectrum does not consist of real eigenvalues. Hence, one cannot directly apply the Mellin transform to the function $g(t) := \text{Tr}(D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2})$. We will use equation (12) in the proof of the functional equations of the super zeta function $Z^\sigma(s; \sigma, \chi)$, where the eta invariant $\eta(0, D_\chi^\sharp(\sigma))$ of $D_\chi^\sharp(\sigma)$ occurs.

The meromorphic continuation of the zeta functions and their functional equations, as they are stated in Theorems H-Q, are the focus of Chapters 6 and 7. In order to prove the meromorphic continuation of the Ruelle zeta function, we prove the representation of the Ruelle zeta function as a product of Selberg zeta functions with twisted origins (cf. Proposition 6.10). For $\sigma \in \widehat{M}$ we define

$$Z_p(s; \sigma, \chi) := \prod_{(\psi_p, \lambda) \in J_p} Z(s + \rho - \lambda; \psi_p \otimes \sigma, \chi). \quad (13)$$

Then,

$$R(s; \sigma, \chi) = \prod_{p=0}^{d-1} Z_p(s; \sigma, \chi)^{(-1)^p}. \quad (14)$$

This expression of the Ruelle zeta function will be used to derive also its functional equations (8), (9), and (10).

In Chapter 8, we define the generalized zeta function $\zeta(z, s; \sigma)$ associated with the operator $A_\chi^\sharp(\sigma)$:

$$\zeta(z, s; \sigma) = \frac{1}{\Gamma(z)} \int_0^\infty e^{-ts^2} \text{Tr} e^{-tA_\chi^\sharp(\sigma)} t^{z-1} dt,$$

for $\text{Re}(s^2) > 0$, $\text{Re}(\lambda_i) > 0$, as well as the regularized determinant of the operator $A_\chi^\sharp(\sigma) + s^2$:

$$\det(A_\chi^\sharp(\sigma) + s^2) := \exp \left(- \frac{d}{dz} \zeta(z, s; \sigma) \Big|_{z=0} \right).$$

We prove the determinant formula, which relates the Selberg zeta function to the regularized determinant of the operator $A_\chi^\sharp(\sigma) + s^2$.

Theorem R. *Let $\det(A_\chi^\sharp(\sigma) + s^2)$ be the regularized determinant associated to the operator $A_\chi^\sharp(\sigma) + s^2$. Then*

1. **case(a)** *the Selberg zeta function has the representation*

$$Z(s; \sigma, \chi) = \det(A_\chi^\sharp(\sigma) + s^2) \exp \left(- 2\pi \dim(V_\chi) \text{Vol}(X) \int_0^s P_\sigma(t) dt \right). \quad (15)$$

2. **case(b)** the symmetrized zeta function has the representation

$$S(s; \sigma, \chi) = \det(A_\chi^\sharp(\sigma) + s^2) \exp\left(-4\pi \dim(V_\chi) \text{Vol}(X) \int_0^s P_\sigma(t) dt\right). \quad (16)$$

We also prove a determinant formula for the Ruelle zeta function.

Proposition S. *The Ruelle zeta function has the representation*

• **case (a)**

$$R(s; \sigma, \chi) = \prod_{p=0}^d \det(A_\chi^\sharp(\sigma_p \otimes \sigma) + (s + \rho - \lambda)^2)^{(-1)^p} \exp\left(-2\pi(d+1) \dim(V_\chi) \dim(V_\sigma) \text{Vol}(X)s\right). \quad (17)$$

• **case (b)**

$$R(s; \sigma, \chi)R(s; w\sigma, \chi) = \prod_{p=0}^d \det(A_\chi^\sharp(\sigma_p \otimes \sigma) + (s + \rho - \lambda)^2)^{(-1)^p} \exp\left(-4\pi(d+1) \dim(V_\chi) \dim(V_\sigma) \text{Vol}(X)s\right). \quad (18)$$

In Chapter 9, we discuss how we want to approach the answer to Question 2, i.e., the generalization of Wotzke's theorem for an arbitrary representation χ of Γ . We consider the flat Laplacian $\Delta_{\chi,p}^\sharp$ acting on p -differential forms on X with values in the flat vector bundle E_χ . We follow [BK05] to define the complex valued analytic torsion $T^\mathbb{C}(\chi; E_\chi)$ associated with $\Delta_{\chi,p}^\sharp$. We want to relate the analytic torsion $T^\mathbb{C}(\chi; E_\chi)$ to the Ruelle zeta function evaluated at 0. We mention the main problems in proving this conjecture. Specifically, the flat Laplacian is not a self-adjoint operator and this causes several problems. We consider an acyclic representation χ of Γ , but we can not apply the Hodge theory to relate the cohomology groups $H^p(X; E_\chi)$ to the kernels $\mathcal{H}^p(X, E_\chi) := \ker(\Delta_{\chi,p})$, for $p = 0, \dots, d$, i.e.

$$H^p(X; E_\chi) \not\cong \mathcal{H}^p(X, E_\chi).$$

Hence, the regularity of the Ruelle zeta function at zero can not be implied.

Last, we include two appendices. In Appendix A, we recall the spectral properties of general elliptic differential operators and define the corresponding spectral ζ -functions and regularized determinants as well. Next, we define the operators

$D_\chi^\sharp(\sigma)$ and $A_\chi^\sharp(\sigma)$ in view of the previous general setting and examine their spectral properties. Our main sources are [Shu87] and [BK08].

Appendix B gives a more detailed introduction into the basic theory and constructions, concerning the heat equation on a compact riemannian manifold X (without boundary). Appendix B is taken from [Mül12a]. It contains detailed proofs of the analytic properties of the heat operator $e^{-t\Delta}$, induced by an elliptic self-adjoint positive differential operator Δ .

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CHAPTER 1

Preliminaries

1.1 Compact hyperbolic odd dimensional manifolds

In this section we will fix notation and give the definitions, which are needed to study the compact hyperbolic odd dimensional manifolds.

Let $G = \text{Spin}(d, 1)$ and $K = \text{Spin}(d)$ or $G = \text{SO}^0(d, 1)$ and $K = \text{SO}(d)$, for $d = 2n + 1$, $n \in \mathbb{N}$. Then, K is a maximal compact subgroup of G . Let $\mathfrak{g}, \mathfrak{k}$ be the Lie algebras of G and K , respectively. We denote by Θ the Cartan involution of G and θ the differential of Θ at $e_G = e$, the identity element of G . It holds $\theta^2 = \text{Id}_{\mathfrak{g}}$. Hence, there exist subspaces \mathfrak{p} and \mathfrak{k} of \mathfrak{g} , such that \mathfrak{p} is the eigenspace for the (-1) -eigenvalue and \mathfrak{k} is the eigenspace for the $(+1)$ -eigenvalue of θ . The Cartan decomposition of \mathfrak{g} is given by

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}. \quad (1.1)$$

We have

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}.$$

Let \mathfrak{a} be a Cartan subalgebra of \mathfrak{p} , i.e. a maximal abelian subalgebra of \mathfrak{p} . We consider the subgroup A of G with Lie algebra \mathfrak{a} . Let M be the centralizer of A in K . Then, $M = \text{Spin}(d - 1)$ or $\text{SO}(d - 1)$. Let \mathfrak{m} be its Lie algebra. Let \mathfrak{b} be a Cartan subalgebra of \mathfrak{m} and \mathfrak{h} a Cartan subalgebra of \mathfrak{g} .

We consider the complexifications

$$\begin{aligned} \mathfrak{g}_{\mathbb{C}} &:= \mathfrak{g} \oplus i\mathfrak{g} \\ \mathfrak{h}_{\mathbb{C}} &:= \mathfrak{h} \oplus i\mathfrak{h} \\ \mathfrak{m}_{\mathbb{C}} &:= \mathfrak{m} \oplus i\mathfrak{m}. \end{aligned}$$

We want to use the theorem of the highest weight for the groups K and M . We recall here some basic facts from the representation theory of compact reductive Lie groups.

Let (Φ, V) be a finite dimensional complex representation of a compact linear connected reductive group \mathcal{K} , with Lie algebra $\mathfrak{g}_{\mathcal{K}}$. By [Kna86, Proposition 1.6], we can regard Φ as unitary. Let ϕ be the differential of Φ at e . Then, $\phi(Y)$ is skew symmetric for every $Y \in \mathfrak{g}_{\mathcal{K}}$. Let \mathfrak{j} be a Cartan subalgebra of $\mathfrak{g}_{\mathcal{K}}$. Let $(H_i), i = 1, \dots, N$ be a basis for \mathfrak{j} . The matrices $\phi(H_i)$ are diagonalizable with imaginary eigenvalues, and since H_i commute with each other, so do $\phi(H_i)$. Hence, there exists a simultaneous eigenspace decomposition of V under $\phi(H_i)$, which can be extended to an eigenspace decomposition under $\phi(\mathfrak{j}_{\mathbb{C}})$, where $\mathfrak{j}_{\mathbb{C}}$ is the complexified algebra of \mathfrak{j} (with real eigenvalues).

Definition 1.1. A weight $\lambda(H) \in (\mathfrak{j}_{\mathbb{C}})^*$ of the representation ϕ is a linear functional on $\mathfrak{j}_{\mathbb{C}}$ such that

$$\phi(H)v = \lambda(H)v,$$

where $v \in V$ with $v \neq 0$.

A weight space V_{λ} is a subspace of V , which is an eigenspace for the eigenvalue $\lambda(H)$, i.e.

$$V_{\lambda} := \{v \in V : \phi(H)v = \lambda(H)v, \forall H \in \mathfrak{j}_{\mathbb{C}}\}.$$

The weight space decomposition is a decomposition of V , given by the finite sum over the weights

$$V = \bigoplus_{\lambda} V_{\lambda}.$$

Let $(\mathfrak{g}_{\mathcal{K}})_{\mathbb{C}}$ be the complexification of $\mathfrak{g}_{\mathcal{K}}$. We define the trace form on $(\mathfrak{g}_{\mathcal{K}})_{\mathbb{C}} \times (\mathfrak{g}_{\mathcal{K}})_{\mathbb{C}}$ by $B_0(X_1, X_2) := \text{Tr}(X_1 X_2)$. It is a complex-valued symmetric bilinear form. We define an inner product $\langle \cdot, \cdot \rangle$ on $(\mathfrak{j}_{\mathbb{C}})^*$ by

$$\langle \lambda_1, \lambda_2 \rangle := B_0(H_{\lambda_1}, H_{\lambda_2}) = \lambda_1(H_{\lambda_2}) = \lambda_2(H_{\lambda_1}). \quad (1.2)$$

Definition 1.2. We call root a non zero weight α for the representation $\phi = \text{ad} : (\mathfrak{g}_{\mathcal{K}})_{\mathbb{C}} \rightarrow \mathfrak{gl}(\mathfrak{g}_{\mathcal{K}})$. The corresponding root space decomposition is given by

$$(\mathfrak{g}_{\mathcal{K}})_{\mathbb{C}} = (\mathfrak{g}_{\mathcal{K}})_0 \oplus \sum_{\alpha \in \Delta((\mathfrak{g}_{\mathcal{K}})_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})} (\mathfrak{g}_{\mathcal{K}})_{\alpha},$$

where $\Delta((\mathfrak{g}_{\mathcal{K}})_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ denotes the set of roots for the adjoint representation of $(\mathfrak{g}_{\mathcal{K}})_{\mathbb{C}}$.

Definition 1.3. Let (H_1, \dots, H_k) be a basis for \mathfrak{j} . We say that $\alpha \in (\mathfrak{j}_{\mathbb{C}})^*$ is positive if it is real on \mathfrak{j} and $\alpha = \alpha(H_i) > 0$, where $i = 1, \dots, r$, and $\alpha(H_j) = 0$ for all $j = 1, \dots, r - 1$.

We write $\alpha > 0$ for a positive root α , and $\alpha_1 > \alpha_2$, if $\alpha_1 - \alpha_2 > 0$. We denote the set of the positive roots by $\Delta^+((\mathfrak{g}_{\mathbb{K}})_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$.

Theorem 1.4 (The theorem of highest weight). *Let \mathcal{K} be a compact linear connected reductive group. Apart from equivalence, the irreducible representations Φ of \mathcal{K} stand in one-to-one correspondence with the highest weights $\lambda \in (\mathfrak{j}_{\mathbb{C}})^*$ (largest weight in the ordering) of Φ_{λ} .*

Proof. See [Kna86, Theorem 4.28]. □

We turn now to the case, where $G = \text{Spin}(d, 1)$, \mathfrak{g} denotes its Lie algebra, and \mathfrak{p} and \mathfrak{a} are as in the beginning of this section. Let (X_1, X_2) be the inner product on $\mathfrak{g} \times \mathfrak{g}$, defined by

$$(X_1, X_2) := -\text{Re}(B_0(X_1, \theta X_2)). \quad (1.3)$$

The adjoint operator $\text{ad}(\mathfrak{p})$ is a symmetric operator on \mathfrak{g} with respect to the inner product (1.2). Hence, if we restrict ad to \mathfrak{a} , we get a commuting family of symmetric transformations on \mathfrak{g} , which can be simultaneously diagonalized. Let $\Delta(\mathfrak{g}, \mathfrak{a})$, $\Delta^+(\mathfrak{g}, \mathfrak{a}) \subset \Delta(\mathfrak{g}, \mathfrak{a})$ be the sets of the restricted, respectively, positive restricted roots of the system $(\mathfrak{g}, \mathfrak{a})$. We define

$$\mathfrak{n} := \bigoplus_{\lambda \in \Delta^+(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}_{\lambda}.$$

Proposition 1.5. *Let \mathfrak{g} be the Lie algebra of G . Then, \mathfrak{g} decomposes as*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n},$$

where \mathfrak{a} is an abelian subalgebra, \mathfrak{n} is nilpotent, $\mathfrak{a} \oplus \mathfrak{n}$ is solvable, and $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] = \mathfrak{n}$.

Proof. See [Kna86, Proposition 5.10]. □

Theorem 1.6 (Iwasawa decomposition of the Lie group G). *Let A and N be the analytic subgroups of G with Lie algebras \mathfrak{a} and \mathfrak{n} . Then, A , N , and AN are simply connected closed subgroups of G , and the multiplication map $K \times A \times N \rightarrow G$, given by $(k, a, n) \rightarrow kan$, is a diffeomorphism onto.*

Proof. See [Kna86, Theorem 5.12]. □

Let $C(X, Y) := \text{Re}(B_0(X, Y))$ be the real part of the trace form on $\mathfrak{g} \times \mathfrak{g}$. We choose a basis (X_i) for \mathfrak{g} and set $c_{ij} = C(X_i, X_j)$. Then, since $C(\cdot, \cdot)$ is a non-degenerate form, the matrix $C := (c_{ij})$ is non-singular. We denote the inverse matrix of C by $C^{-1} = (c^{ij}) := (c_{ij})^{-1}$. We put $X^j = \sum c^{ij} X_i$, so that $X_i = \sum c_{ji} X^j$. Let $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ be the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$.

Definition 1.7. We define the Casimir element $\Omega \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ by

$$\Omega := \sum_{i,j} X^i X^j.$$

Proposition 1.8. *The Casimir element Ω is independent of the basis (X_i) . Furthermore, it satisfies $\text{Ad}(g)\Omega = \Omega$ for all $g \in G$ and hence is in the center $Z(\mathfrak{g}_{\mathbb{C}})$ of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$.*

Proof. This is proved in [Kna86, Proposition 8.6]. \square

Let (\cdot, \cdot) be the inner product on \mathfrak{g} , defined by (1.3). Let (X_i) be an orthonormal basis of \mathfrak{p} and (Y_j) an orthonormal basis of \mathfrak{k} , with respect to this inner product. By Definition 1.7 we have

$$\begin{aligned} \Omega &= \sum_i X_i^2 - \sum_j Y_j^2 \\ \Omega_K &= - \sum_j Y_j^2. \end{aligned}$$

Here Ω_K denotes the Casimir element, which corresponds to the restriction $(\cdot, \cdot)|_{\mathfrak{k} \times \mathfrak{k}}$. It lies in the center $Z(\mathfrak{k})$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{k})$ of \mathfrak{k} .

If we consider a finite dimensional unitary irreducible representation (τ, V_τ) of K , then, since $\Omega \in Z(\mathfrak{k})$, Schur's Lemma (cf. [Kna86, Proposition 1.5]) assures us that $\tau(\Omega_K)$ acts by a scalar λ_τ , called the Casimir eigenvalue of τ . Then,

$$\tau(\Omega_K) = \lambda_\tau \text{Id}_{V_\tau}.$$

The group $\text{SO}^0(d, 1)$ acts transitively on the hyperbolic space:

$$\mathbb{H}^d = \{(x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1} : x_1^2 - x_2^2 \dots - x_{d+1}^2 = 1, x_1 > 0\}.$$

The stabilizer of the point $(1, 0, \dots, 0)$ is $\text{SO}(d)$, which is a maximal compact subgroup of $\text{SO}^0(d, 1)$. G and K are the universal covering groups of $\text{SO}^0(d, 1)$ and $\text{SO}(d)$, respectively. We set

$$\tilde{X} := G/K.$$

Let $T_{eK}\tilde{X}$ be the tangent space of \tilde{X} at $eK \in \tilde{X}$. There is a canonical isomorphism

$$T_{eK}\tilde{X} \cong \mathfrak{p}.$$

Let $B(X, Y)$ be the Killing form on $\mathfrak{g} \times \mathfrak{g}$ defined by

$$B(X, Y) = \text{Tr}(\text{ad}(X) \circ \text{ad}(Y)).$$

It is a symmetric bilinear form. We consider the inner product $\langle \cdot, \cdot \rangle_0$, induced by the Killing form

$$\langle Y_1, Y_2 \rangle_0 := \frac{1}{2(d-1)} B(Y_1, Y_2), \quad Y_1, Y_2 \in \mathfrak{g}. \quad (1.4)$$

The restriction of $\langle \cdot, \cdot \rangle_0$ to \mathfrak{p} satisfies

$$\langle \text{Ad}(k)(Y_1), \text{Ad}(k)(Y_2) \rangle_0 = \langle Y_1, Y_2 \rangle_0, \quad Y_1, Y_2 \in \mathfrak{p}, k \in K. \quad (1.5)$$

We consider now the left translation $L_g(x) = gx, g \in G, x \in \tilde{X}$. We define a riemannian metric on \tilde{X} by the inner product

$$\langle X_1, X_2 \rangle = \langle dL_{g^{-1}}(X_1), dL_{g^{-1}}(X_2) \rangle_0, \quad X_1, X_2 \in T_x \tilde{X}. \quad (1.6)$$

By (1.5) the right hand side of (1.6) is independent of the chosen representative of the coset gK . Therefore, the riemannian metric is G -invariant. Then,

$$\tilde{X} \cong \mathbb{H}^d.$$

Let $\Gamma \subset G$ be a discrete torsion-free cocompact subgroup of G . Then,

$$X := \Gamma \backslash G / K = \Gamma \backslash \tilde{X}$$

is a compact hyperbolic manifold of dimension d , with universal covering \tilde{X} . We equip X with the riemannian metric, induced by the inner product (1.4). Then, X has constant negative sectional curvature -1 .

Let $\Delta^+(\mathfrak{g}, \mathfrak{a})$ be the set of positive roots of the system $(\mathfrak{g}, \mathfrak{a})$. Then, $\Delta^+(\mathfrak{g}, \mathfrak{a}) = \{\alpha\}$. Let $M' = \text{Norm}_K(A)$. We define the restricted Weyl group (analytically) as the quotient

$$W_A := M' / M.$$

Then, W_A has order 2.

Let $H_{\mathbb{R}} \in \mathfrak{a}$ such that $\alpha(H_{\mathbb{R}}) = 1$. With respect to the inner product (1.3), $H_{\mathbb{R}}$ has norm 1. We define

$$A^+ := \{\exp(tH_{\mathbb{R}}) : t \in \mathbb{R}^+\}. \quad (1.7)$$

We define also

$$\rho := \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})} \dim(\mathfrak{g}_{\alpha}) \alpha, \quad (1.8)$$

$$\rho_{\mathfrak{m}} := \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b})} \alpha. \quad (1.9)$$

The inclusion $i: M \hookrightarrow K$ induces the restriction map $i^*: R(K) \rightarrow R(M)$, where $R(K), R(M)$ are the representation rings over \mathbb{Z} of K and M , respectively. Let \widehat{K}, \widehat{M} be the sets of equivalent classes of irreducible unitary representations of K and M , respectively. By the theorem of the highest weight (Theorem 1.4), the representations $\tau \in \widehat{K}, \sigma \in \widehat{M}$ are parametrized by their highest weights ν_τ, ν_σ , respectively. Then,

$$\nu_\tau = (\nu_1, \dots, \nu_n),$$

where $\nu_1 \geq \dots \geq \nu_n$ and $\nu_i, i = 1, \dots, n$ are all integers or all half integers (that is $\nu_i = q_i/2, q_i \in \mathbb{Z}$) and

$$\nu_\sigma = (\nu_1, \dots, \nu_{n-1}, \nu_n), \quad (1.10)$$

where $\nu_1 \geq \dots \geq \nu_{n-1} \geq |\nu_n|$ and $\nu_i, i = 1, \dots, n$ are all integers or all half integers. Let s be the spin representation of K , given by

$$s: K \rightarrow \text{End}(\Delta_{2n}) \oplus \text{End}(\Delta_{2n}) \xrightarrow{pr} \text{End}(\Delta_{2n})$$

where $\Delta_{2n} := \mathbb{C}^{2^k}$ such that $n = k$, and pr denotes the projection onto the first component (cf. [Fri00, p.14]). We set for abbreviation $S = \Delta_{2n}$. Let $(s^+, S^+), (s^-, S^-)$ be the half spin representations of M , where $S^\pm := \Delta^\pm$ (cf. [Fri00, p.22]). The highest weight of s is given by

$$\nu_s = \left(\frac{1}{2}, \dots, \frac{1}{2}\right),$$

and the highest weights of s^+, s^- are

$$\nu_{s^+} = \left(\frac{1}{2}, \dots, \frac{1}{2}\right) \quad (1.11)$$

$$\nu_{s^-} = \left(\frac{1}{2}, \dots, -\frac{1}{2}\right), \quad (1.12)$$

respectively. Let $w \in W_A$ be a non trivial element of W_A , and m_w a representative of w in M' . Then the action of W_A on \widehat{M} is defined by

$$(w\sigma)(m) := \sigma(m_w^{-1}mm_w), \quad m \in M, \sigma \in \widehat{M}.$$

If $\nu_\sigma = (\nu_1, \dots, \nu_{n-1}, \nu_n)$ is the highest weight of σ , then the highest weight of $w\sigma$ is given by

$$\nu_{w\sigma} = (\nu_1, \dots, \nu_{n-1}, -\nu_n).$$

Specifically, for the half spin representations s^\pm we have

$$\nu_{ws^\pm} = \left(\frac{1}{2}, \dots, \mp \frac{1}{2}\right). \quad (1.13)$$

Hence,

$$ws^\pm = s^\mp. \quad (1.14)$$

1.2 Haar measure on G

We want to define a measure on our Lie group G , using the Iwasawa decomposition. First, we set $a(t) = \exp(tH_{\mathbb{R}}) \in A$, $t \in \mathbb{R}$. Then, we can use a Lebesgue measure on A , induced by the Lebesgue measure on \mathbb{R} . Since K is compact, the Haar measure dk on K can be normalized such that

$$\int_K dk = 1.$$

For the N -component, we deal first with the Lie algebra \mathfrak{n} of N . We consider an isometric identification of \mathbb{R}^{2n} with \mathfrak{n} with respect to the inner product

$$\langle Y_1, Y_2 \rangle_{\theta} := -\frac{1}{2(d-1)} B(Y_1, \theta(Y_2)),$$

where $Y_1, Y_2 \in \mathfrak{n}$. Then, we can equip \mathfrak{n} with the measure induced by the Lebesgue measure on \mathbb{R}^{2n} . We use the exponential map to pass to the Lie group N and define a Haar measure dn on N , induced by the measure on \mathfrak{n} .

Lemma 1.9. *Let $S = AN$. Let da and dn be left invariant measure on A and N , respectively. Then, the left invariant measure ds on S can be normalized such that for $f \in C_0(S)$*

$$\int_S f(s) ds = \int_A \int_N f(an) da dn.$$

Proof. See [Wal73, Lemma 7.6.2]. □

Lemma 1.10. *Let d_{RS} be a right invariant measure on S . Then, for $f \in C_0(S)$,*

$$\int_S f(s) d_{RS} = \int_{\mathbb{R}} \int_N f(a(t)n) e^{\rho t H_{\mathbb{R}}} dn dt,$$

where ρ as in (1.8).

Proof. We consider the modular function δ on S . It is a smooth non-vanishing real-valued function, such that $d_R(s) = \delta(s) ds$ and

$$\delta(s) = \det(\text{Ad}(s)) \tag{1.15}$$

([Wal73, p.31-32]). On the other hand, by Proposition 1.5, \mathfrak{n} is a nilpotent subalgebra of \mathfrak{g} . Therefore, there exists $k \in \mathbb{N}$ such that for $n \in N$, $(\text{Ad}(n) - \text{Id})^k = 0$. Hence,

$$\text{Ad}(n)(\mathfrak{a} + \mathfrak{n}) = \text{Id}|_{\mathfrak{a} + \mathfrak{n}}. \tag{1.16}$$

Also,

$$\text{Ad}(a(t))|_{\mathfrak{a}} = \text{Id}|_{\mathfrak{a}}. \quad (1.17)$$

Equation (1.15) becomes by equation (1.16) and (1.17),

$$\delta(s) = \delta(a(t)n) = \det(\text{Ad}(a(t)n)|_{\mathfrak{a}+\mathfrak{n}}) = \det(\text{Ad}(a(t))|_{\mathfrak{a}+\mathfrak{n}}) = \det(\text{Ad}(a(t))|_{\mathfrak{n}}). \quad (1.18)$$

We use now the definition of ρ (equation (1.8)) and write $\rho(H) = \frac{1}{2} \text{tr}(\text{ad}(H)|_{\mathfrak{n}})$. We have

$$\exp(2\rho(\log a(t))) = \exp(\text{tr ad}(\log a(t))|_{\mathfrak{n}}) = \det(\text{Ad}(a(t))|_{\mathfrak{n}}), \quad (1.19)$$

where we have used the identities

$$\begin{aligned} \text{Ad}(\exp(X)) &= \exp(\text{ad}(X)), \\ \det(\text{Ad}(\exp(X))) &= \exp(\text{tr}(\text{ad}(X))) \end{aligned}$$

for $X \in \mathfrak{g}$. Hence, by equations (1.18) and (1.19),

$$\delta(a(t)n) = \exp(2\rho t H_{\mathbb{R}}).$$

The assertion follows from Lemma 1.9. \square

Proposition 1.11. *The invariant measure dg on G can be normalized such that for $f \in C_0(G)$,*

$$\int_G f(g) d_R g = \int_K \int_{\mathbb{R}} \int_N f(ka(t)n) e^{\rho t H_{\mathbb{R}}} dn dt dk. \quad (1.20)$$

Proof. Let d_R be a right invariant measure on S . Let β be a function defined as $\beta : K \times S \rightarrow G$, $(k, s) \mapsto ks$. Then, $\beta^* dg = h(k, s) dk d_R s$. We pick an element $k_0 \in K$ and consider the left action L_{k_0} in the first component $L_{k_0} \beta(k, s) = \beta(k_0 k, s)$. Then, since K is unimodular,

$$(L_{k_0} \beta(k, s))^* dg = (\beta(k_0 k, s))^* dg = h(k_0 k, s) dk d_R s. \quad (1.21)$$

On the other hand, since G is unimodular,

$$(L_{k_0} \beta(k, s))^* dg = \beta(k, s)^* L_{k_0} dg = \beta(k, s)^* dg = h(k, s) dk d_R s. \quad (1.22)$$

Hence, by equation (1.21), and (1.22), we get

$$h(k_0 k, s) = h(k, s), \quad \forall k_0, k \in K, s \in S. \quad (1.23)$$

Similarly, we pick an element $s_0 \in S$, and we consider the right action R_{s_0} in the second component $R_{s_0}\beta(k, s) = \beta(k, ss_0)$. We have

$$(R_{s_0}\beta(k, s))^*dg = (\beta(k, ss_0))^*dg = h(k, ss_0)dkd_{RS}. \quad (1.24)$$

Since G is unimodular,

$$(R_{s_0}\beta(k, s))^*dg = \beta(k, s)^*R_{s_0}dg = \beta(k, s)^*dg = h(k, s)dkd_{RS}. \quad (1.25)$$

So, by equations (1.24) and (1.25) we obtain

$$h(k, ss_0) = h(k, s), \quad \forall k \in K, s, s_0 \in S. \quad (1.26)$$

Using now equations (1.23) and (1.26), we conclude that $h(k, s)$ is a constant function on $K \times S$. The assertion follows from Lemma 1.10. \square

We define now a left invariant measure on the the quotient space $\Gamma \backslash G$. Let $p: G \rightarrow \Gamma \backslash G$ be the projection map. We define the map $J: C_c(G, \mathbb{C}) \rightarrow C_c(\Gamma \backslash G, \mathbb{C})$, given by

$$(Jf)(p(g)) := \sum_{\gamma \in \Gamma} f(\gamma g). \quad (1.27)$$

Proposition 1.12. *There exists a left invariant Haar measure dx on the quotient space $\Gamma \backslash G$, which can be normalized such that for every $f \in C_c(G, \mathbb{C})$,*

$$\int_{\Gamma \backslash G} (Jf)(x)dx = \int_G f(g)dg. \quad (1.28)$$

Proof. See [Bum97, Proposition 4.3.5]. \square

Remark 1.13. *The same setting can be considered for defining a Haar measure $d\tilde{x}$ on $\tilde{X} = G/K$. Let $\pi: G \rightarrow G/K$ be the projection map. We define a surjective map $I: C_c(G, \mathbb{C}) \rightarrow C_c(G/K, \mathbb{C})$ by*

$$(If)(\pi(g)) := \int_K f(gk)dk.$$

Then,

$$\int_{G/K} (If)(\tilde{x})d\tilde{x} = \int_G f(g)dg. \quad (1.29)$$

1.3 Word metric

For the proof of the convergence of the Ruelle and Selberg zeta functions we need the word metric on Γ , so that we can obtain an upper bound for the character of the representation of Γ .

Let Γ be a finitely generated group with unite element e . Let $L = \{a_1, \dots, a_k\}$ be a set of generators . Let $L^{-1} = \{a_1^{-1}, \dots, a_k^{-1}\}$ be the set of the inverse elements of L . Then, every element $g \neq e$ in Γ can be written as

$$g = a_1^{\epsilon_1} \cdots a_r^{\epsilon_r},$$

where $\epsilon_i \in \mathbb{Z}$, $1 \leq i \leq r$, and $r \leq k$.

Definition 1.14. The length of a non trivial element $g \in \Gamma$ is defined to be the minimal positive integer $l \in \mathbb{N}$ such that g can be written as a product of l -elements of $L \cup L^{-1}$, counted with multiplicity. The length of $e \in \Gamma$ is defined to be 0.

If g has length l , then we say that g can be written as word of length l .

Definition 1.15. The word metric on Γ is defined to be

$$d_W(g, g') = l, \quad g, g' \in \Gamma,$$

where l is the length of $g^{-1}g'$.

We consider the action of a discrete torsion-free cocompact subgroup Γ of $G = \text{Spin}(d, 1)$ on the symmetric space $\tilde{X} = G/K = \text{Spin}(d, 1)/\text{Spin}(d)$. We define a word metric d_W on Γ . The fact that Γ is cocompact assures us that the riemannian metric on G/K , restricted to Γx_0 for $x_0 \in \tilde{X}$, is Lipschitz equivalent to d_W .

Proposition 1.16. *Let Γ be a discrete torsion-free cocompact subgroup of G . Let d_W be a fixed left invariant word metric on Γ . We can embed Γ in \tilde{X} via the map $\Gamma \rightarrow \Gamma x_0$, $x_0 \in \tilde{X}$. Then, the pullback of the restriction of the riemannian metric d on \tilde{X} to Γx_0 is Lipschitz equivalent to d_W .*

Proof. This is proved in [LMR00, Prop. 3.2]. □

CHAPTER 2

Dynamical zeta functions

2.1 Twisted Ruelle and Selberg zeta function

Throughout this chapter we will consider finite dimensional representations $\chi: \Gamma \rightarrow \mathrm{GL}(V_\chi)$ of Γ , which are not necessarily unitary.

Definition 2.1. Let $\sigma \in \widehat{M}$. The twisted Selberg zeta function $Z(s; \sigma, \chi)$ for X is defined by the infinite product

$$Z(s; \sigma, \chi) := \prod_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \prod_{k=0}^{\infty} \det \left(\mathrm{Id} - (\chi(\gamma) \otimes \sigma(m_\gamma) \otimes S^k(\mathrm{Ad}(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}})) \right) e^{-(s+\rho)l(\gamma)}, \quad (2.1)$$

where $s \in \mathbb{C}$, $\bar{\mathfrak{n}} = \theta \mathfrak{n}$ is the sum of the negative root spaces of \mathfrak{a} , $S^k(\mathrm{Ad}(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}})$ denotes the k -th symmetric power of the adjoint map $\mathrm{Ad}(m_\gamma a_\gamma)$ restricted to $\bar{\mathfrak{n}}$, and ρ is as in (1.8).

Definition 2.2. Let $\sigma \in \widehat{M}$. The twisted Ruelle zeta function $R(s; \sigma, \chi)$ for X is defined by the infinite product

$$R(s; \sigma, \chi) := \prod_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \det \left(\mathrm{Id} - \chi(\gamma) \otimes \sigma(m_\gamma) e^{-sl(\gamma)} \right)^{(-1)^{d-1}}. \quad (2.2)$$

2.2 Convergence of the zeta functions

In Proposition 1.16, we have seen the equivalence of the word metric and the riemannian metric on Γ -orbits in \tilde{X} . This fact will be used in the proof of Lemma 2.3 below to find an upper bound for the character of any finite dimensional representation of Γ . Furthermore, we will use the definition of the length of γ with respect to the word metric d_W . We define this length by

$$l_W(\gamma) := d_W(\gamma, e).$$

Lemma 2.3. *Let $\chi: \Gamma \rightarrow \text{GL}(V_\chi)$ be a finite dimensional representation of Γ . Then, there exist positive constants $K, k > 0$ such that*

$$|\text{tr}(\chi(\gamma))| \leq K e^{kl(\gamma)}, \quad \forall \gamma \in \Gamma - \{e\}. \quad (2.3)$$

Proof. We fix a finite set of generators $L = \{\gamma_1, \dots, \gamma_r\}$ of Γ , and we choose a norm $\|\cdot\|$ on V_χ . Then, if we put $C = \max\{\|\chi(\gamma_i)\| : \gamma_i \in L \cup L^{-1}\}$, we get for $c = \log C$,

$$\|\chi(\gamma)\| \leq e^{cl_W(\gamma)}. \quad (2.4)$$

By Proposition 1.16, it follows that there exist positive constants $c_1, c_2 > 0$ such that

$$c_1 d(x_0, \gamma x_0) \leq d_W(e, \gamma) \leq c_2 d(x_0, \gamma x_0), \quad (2.5)$$

where $x_0 := eK$ is the identity element of $\tilde{X} = G/K$.

Then, (2.4) becomes by (2.5)

$$\|\chi(\gamma)\| \leq C_1 e^{c_2 d(x_0, \gamma x_0)}.$$

It follows that

$$|\text{tr} \chi(\gamma)| \leq \dim(V_\chi) \|\chi(\gamma)\| \leq C_3 e^{c_2 d(x_0, \gamma x_0)}. \quad (2.6)$$

Now by definition

$$l(\gamma) := \min\{d(x, \gamma x) : x \in \tilde{X}\}.$$

We choose a fundamental domain $F \subset \tilde{X}$ for Γ such that $x_0 \in F$. Given $\gamma \in \Gamma$, let x_1 be in \tilde{X} , such that $l(\gamma) = d(x_1, \gamma x_1)$. Then, there exists a $\gamma_1 \in \Gamma$ such that $x_1 \in \gamma_1 F$. Let $x_2 \in F$ such that $x_1 = \gamma_1 x_2$. By compactness of the fundamental domain, $\text{diam}(F)$ is finite. If we put $\delta := \text{diam}(F)$, then

$$d(x_0, x_2) \leq \delta. \quad (2.7)$$

We see that

$$\begin{aligned} d(x_0, \gamma_1^{-1} \gamma \gamma_1 x_0) &\leq d(x_0, x_2) + d(x_2, \gamma_1^{-1} \gamma \gamma_1 x_0) \\ &\leq \delta + d(x_2, \gamma_1^{-1} \gamma \gamma_1 x_0). \end{aligned} \quad (2.8)$$

In addition,

$$\begin{aligned}
d(x_2, \gamma_1^{-1} \gamma \gamma_1 x_0) &\leq d(x_2, \gamma_1^{-1} \gamma \gamma_1 x_2) + d(\gamma_1^{-1} \gamma \gamma_1 x_2, \gamma_1^{-1} \gamma \gamma_1 x_0) \\
&\leq d(x_2, \gamma_1^{-1} \gamma \gamma_1 x_2) + d(x_0, x_2) \\
&\leq d(x_2, \gamma_1^{-1} \gamma \gamma_1 x_2) + \delta.
\end{aligned} \tag{2.9}$$

Hence, by (2.8) and (2.9) we get

$$d(x_0, \gamma_1^{-1} \gamma \gamma_1 x_0) \leq 2\delta + d(x_2, \gamma_1^{-1} \gamma \gamma_1 x_2). \tag{2.10}$$

Recall that $x_1 = \gamma_1 x_2$. Therefore, we have

$$\begin{aligned}
d(x_0, \gamma_1^{-1} \gamma \gamma_1 x_0) &\leq 2\delta + d(\gamma_1^{-1} x_1, \gamma_1^{-1} \gamma x_1) \\
&\leq 2\delta + d(x_1, \gamma x_1).
\end{aligned} \tag{2.11}$$

Using (2.6), we obtain the following inequalities.

$$\begin{aligned}
|\mathrm{tr}(\chi(\gamma))| &= |\mathrm{tr}(\chi(\gamma_1^{-1} \gamma \gamma_1))| \\
&\leq C_3 e^{c_2 d(x_0, \gamma_1^{-1} \gamma \gamma_1 x_0)} \\
&\leq C_3 e^{c_2 (2\delta + d(x_1, \gamma x_1))} \\
&= C_4 e^{c_2 d(x_1, \gamma x_1)} = C_4 e^{c_2 l(\gamma)}.
\end{aligned}$$

The assertion follows. □

We are ready now to prove the convergence of Selberg and Ruelle zeta functions.

Proposition 2.4. *Let $\chi: \Gamma \rightarrow \mathrm{GL}(V_\chi)$ be a finite dimensional representation of Γ and $\sigma \in \widehat{M}$. Then there exists a constant $c > 0$ such that*

$$Z(s; \sigma, \chi) := \prod_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \prod_{k=0}^{\infty} \det(1 - (\chi(\gamma) \otimes \sigma(m_\gamma) \otimes S^k(\mathrm{Ad}(m_\gamma a_\gamma)_{\bar{\pi}}))) e^{-(s+|\rho|)l(\gamma)}, \tag{2.12}$$

converges absolutely and uniformly on compact subsets of the half-plane $\mathrm{Re}(s) > c$.

Proof. We observe that

$$\begin{aligned}
\log Z(s; \sigma, \chi) &= \sum_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \sum_{k=0}^{\infty} \operatorname{tr} \log(1 - (\chi(\gamma) \otimes \sigma(m_\gamma) \otimes S^k(\operatorname{Ad}(m_\gamma a_\gamma)_{\bar{\mathbb{N}}})) e^{-(s+|\rho)l(\gamma)}) \\
&= - \sum_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \frac{\operatorname{tr}((\chi(\gamma) \otimes \sigma(m_\gamma) \otimes S^k(\operatorname{Ad}(m_\gamma a_\gamma)_{\bar{\mathbb{N}}})) e^{-(s+|\rho)l(\gamma)})^j}{j} \\
&= - \sum_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \sum_{k=0}^{\infty} \frac{1}{n_\Gamma(\gamma)} \operatorname{tr}(\chi(\gamma) \otimes \sigma(m_\gamma) \otimes S^k(\operatorname{Ad}(m_\gamma a_\gamma)_{\bar{\mathbb{N}}})) e^{-(s+|\rho)l(\gamma)} \\
&= - \sum_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \frac{1}{n_\Gamma(\gamma)} \operatorname{tr}(\chi(\gamma) \otimes \sigma(m_\gamma)) \frac{e^{-(s+|\rho)l(\gamma)}}{\det(\operatorname{Id} - \operatorname{Ad}(m_\gamma a_\gamma)_{\bar{\mathbb{N}}})},
\end{aligned} \tag{2.13}$$

where in the last equation, we made use of the identity

$$\sum_{k=0}^{\infty} S^k(\operatorname{Ad}(m_\gamma a_\gamma)_{\bar{\mathbb{N}}}) = \frac{1}{\det(\operatorname{Id} - \operatorname{Ad}(m_\gamma a_\gamma)_{\bar{\mathbb{N}}})}.$$

Initially, we observe that

$$|\operatorname{tr} \sigma(m_\gamma)| \leq \dim(\sigma), \quad \forall \sigma \in \widehat{M}. \tag{2.14}$$

We need an upper bound for the growth of the length spectrum $l(\gamma)$. By Proposition 1.11 we see that by this normalization of the Haar measure on G , there exists a positive constant $C > 0$ such that for every $R > 0$

$$\operatorname{Vol}(B(x_0, R)) \leq C e^{2|\rho|R},$$

where ρ as in (1.8). Since Γ is a cocompact lattice of G , there exists a positive constant C' , such that

$$\#\{[\gamma] : l(\gamma) < R\} \leq \#\{\gamma \in \Gamma : l(\gamma) \leq R\} \leq C' e^{2|\rho|R}. \tag{2.15}$$

We need also an upper bound for the quantity

$$\frac{1}{\det(\operatorname{Id} - \operatorname{Ad}(m_\gamma a_\gamma)_{\bar{\mathbb{N}}})}.$$

By equation (1.19) in Section 1.2 we have that

$$\det(\operatorname{Ad}(a_\gamma)_{\bar{\mathbb{N}}}) = \exp(-2|\rho|l(\gamma)).$$

We use the estimates (2.15) to see that we can consider a $[\gamma_{min}]$ among all the conjugacy classes of Γ such that $l(\gamma_{min})$ is of minimum length. Hence, there exists a positive constant $C''' > 0$ such that

$$\frac{1}{\det(\text{Id} - \text{Ad}(m_\gamma a_\gamma)_{\bar{\pi}})} < C'''. \quad (2.16)$$

By Lemma 2.3, it follows that there exist positive constants $C, c_1 > 0$ such that

$$\begin{aligned} \sum_{[\gamma] \neq e} \frac{1}{n_\Gamma(\gamma)} \left| \text{tr}(\chi(\gamma) \otimes \sigma(m_\gamma)) \frac{e^{-(s+|\rho)l(\gamma)}}{\det(\text{Id} - \text{Ad}(m_\gamma a_\gamma)_{\bar{\pi}})} \right| \\ \leq C \sum_{[\gamma] \neq e} e^{(c_1 - \text{Re}(s))l(\gamma)} \\ = C \sum_{k=0}^{\infty} \sum_{\substack{[\gamma] \neq e \\ k \leq l(\gamma) \leq k+1}} e^{(c_1 - \text{Re}(s))l(\gamma)} \\ \leq C \sum_{k=0}^{\infty} \mathcal{N}(k+1) e^{(c_1 - \text{Re}(s))k}, \end{aligned}$$

where

$$\mathcal{N}(R) := \#\{[\gamma] \in \Gamma : l(\gamma) \leq R\}, \quad R \geq 0.$$

By (2.15), we have

$$\sum_{k=0}^{\infty} \mathcal{N}(k+1) e^{(c_1 - \text{Re}(s))k} \leq C'' \sum_{k=0}^{\infty} e^{(2|\rho| + c_1 - \text{Re}(s))k}. \quad (2.17)$$

Hence, there exists a positive constant $c > 0$ such that for $s \in \mathbb{C}$ with $\text{Re}(s) > c$,

$$\sum_{[\gamma] \neq e} \frac{1}{n_\Gamma(\gamma)} \left| \text{tr}(\chi(\gamma) \otimes \sigma(m_\gamma)) \frac{e^{-(s+|\rho)l(\gamma)}}{\det(\text{Id} - \text{Ad}(m_\gamma a_\gamma)_{\bar{\pi}})} \right| < \infty.$$

□

A similar approach will be used to establish the convergence of the Ruelle zeta function.

Proposition 2.5. *Let $\chi: \Gamma \rightarrow \text{GL}(V_\chi)$ be a finite dimensional representation of Γ and $\sigma \in \widehat{M}$. Then, there exists a constant $r > 0$ such that*

$$R(s; \sigma, \chi) := \prod_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \det(\text{Id} - \chi(\gamma) \otimes \sigma(m_\gamma) e^{-sl(\gamma)})^{(-1)^{d-1}}. \quad (2.18)$$

converges absolutely and uniformly on compact subsets of the half-plane $\text{Re}(s) > r$.

Proof. We observe that

$$\begin{aligned}
\log R(s; \sigma, \chi) &= (-1)^{d-1} \sum_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \operatorname{tr} \log(1 - \chi(\gamma) \otimes \sigma(m_\gamma) e^{-sl(\gamma)}) \\
&= (-1)^d \sum_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \sum_{j=1}^{\infty} \frac{\operatorname{tr}((\chi(\gamma) \otimes \sigma(m_\gamma) e^{-sl(\gamma)})^j)}{j} \\
&= (-1)^d \sum_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \frac{1}{n_\Gamma(\gamma)} \operatorname{tr}(\chi(\gamma) \otimes \sigma(m_\gamma)) e^{-sl(\gamma)}. \tag{2.19}
\end{aligned}$$

By Lemma 2.3, it follows that there exist positive constants $C, c_1 > 0$ such that

$$\begin{aligned}
\sum_{[\gamma] \neq e} \frac{1}{n_\Gamma(\gamma)} \left| \operatorname{tr}(\chi(\gamma) \otimes \sigma(m_\gamma)) e^{-sl(\gamma)} \right| \\
\leq C \sum_{[\gamma] \neq e} e^{(c_1 - \operatorname{Re}(s))l(\gamma)} \\
= C \sum_{k=0}^{\infty} \sum_{\substack{[\gamma] \neq e \\ k \leq l(\gamma) \leq k+1}} \mathcal{N}(k+1) e^{(c_1 - \operatorname{Re}(s))l(\gamma)} \\
\leq C \sum_{k=0}^{\infty} \mathcal{N}(k+1) e^{(c_1 - \operatorname{Re}(s))k}.
\end{aligned}$$

By (2.15), we have

$$\sum_{k=0}^{\infty} \mathcal{N}(k+1) e^{(c_1 - \operatorname{Re}(s))k} \leq C' \sum_{k=0}^{\infty} e^{(2|\rho| + c_1 - \operatorname{Re}(s))k}.$$

Hence, there exists a positive constant $r > 0$ such that for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > r$,

$$\sum_{[\gamma] \neq e} \frac{1}{n_\Gamma(\gamma)} \left| \operatorname{tr}(\chi(\gamma) \otimes \sigma(m_\gamma)) e^{-sl(\gamma)} \right| < \infty. \tag{2.20}$$

□

2.3 The logarithmic derivative of the zeta functions

We distinguish two cases according to the action of the restricted Weyl group W_A :

1. **case(a)** $\sigma \in \widehat{M}$ is invariant under the the action of the Weyl group W_A .
2. **case(b)** $\sigma \in \widehat{M}$ is not invariant under the the action of the Weyl group W_A .

Throughout this thesis, we will always deal the two cases separately. In case (b), we have to define two special kinds of zeta functions.

Definition 2.6. We define the symmetrized zeta function $S(s; \sigma, \chi)$ by

$$S(s; \sigma, \chi) := Z(s; \sigma, \chi)Z(s; w\sigma, \chi), \quad (2.21)$$

where $s \in \mathbb{C}$, and w is a non trivial element of W_A .

Definition 2.7. We define the super zeta function by

$$Z^s(s, \sigma, \chi) := \frac{Z(s; \sigma, \chi)}{Z(s; w\sigma, \chi)}, \quad (2.22)$$

where $s \in \mathbb{C}$, and w is a non-trivial element of W_A .

The logarithmic derivative of the Selberg, symmetrized and super zeta function plays a crucial role in our analysis in order to obtain the meromorphic continuation of these functions and the Ruelle zeta function as well, to the whole complex plane \mathbb{C} .

Lemma 2.8 (Logarithmic derivatives of the zeta functions). *Let*

$$L_{sym}(\gamma; \sigma) := \frac{\text{tr}(\chi(\gamma) \otimes \sigma(m_\gamma))e^{-|\rho|l(\gamma)}}{\det(\text{Id} - \text{Ad}(m_\gamma a_\gamma)_{\bar{n}})}. \quad (2.23)$$

Then we have

1. **case(a)** *The logarithmic derivative of the Selberg zeta function $Z(s; \sigma, \chi)$ is given by*

$$L(s) := \frac{d}{ds} \log(Z(s; \sigma, \chi)) = \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} L_{sym}(\gamma; \sigma) e^{-sl(\gamma)}. \quad (2.24)$$

2. **case(b)** *The logarithmic derivative of the symmetrized zeta function $S(s; \sigma, \chi)$ is given by*

$$L_S(s) := \frac{d}{ds} \log(S(s; \sigma, \chi)) = \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} L_{sym}(\gamma; \sigma + w\sigma) e^{-sl(\gamma)}. \quad (2.25)$$

3. **case(b)** The logarithmic derivative of the super zeta function $Z^s(s; \sigma, \chi)$ is given by

$$L^s(s) := \frac{d}{ds} \log(Z^s(s; \sigma, \chi)) = \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} L_{sym}(\gamma; \sigma - w\sigma) e^{-sl(\gamma)}. \quad (2.26)$$

Proof. 1. For the case (a), we see by equation (2.13)

$$\begin{aligned} \frac{d}{ds} \log(Z(s; \sigma, \chi)) &= \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} \operatorname{tr}(\chi(\gamma) \otimes \sigma(m_\gamma)) \frac{e^{-sl(\gamma)} e^{-|\rho|l(\gamma)}}{\det(1 - \operatorname{Ad}(m_\gamma a_\gamma)_{\bar{\mathbb{N}}})} \\ &= \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} L_{sym}(\gamma; \sigma) e^{-sl(\gamma)}. \end{aligned}$$

2. In case (b), for the symmetrized zeta function $S(s; \sigma, \chi)$, we see by equation (2.21)

$$\begin{aligned} \frac{d}{ds} \log(S(s; \sigma, \chi)) &= \frac{d}{ds} \log(Z(s; \sigma, \chi)) + \frac{d}{ds} \log(Z(s; w\sigma, \chi)) \\ &= \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} \operatorname{tr}(\chi(\gamma) \otimes \sigma(m_\gamma)) \frac{e^{-sl(\gamma)} e^{-|\rho|l(\gamma)}}{\det(1 - \operatorname{Ad}(m_\gamma a_\gamma)_{\bar{\mathbb{N}}})} \\ &\quad + \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} \operatorname{tr}(\chi(\gamma) \otimes w\sigma(m_\gamma)) \frac{e^{-sl(\gamma)} e^{-|\rho|l(\gamma)}}{\det(1 - \operatorname{Ad}(m_\gamma a_\gamma)_{\bar{\mathbb{N}}})} \\ &= \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} L_{sym}(\gamma; \sigma + w\sigma) e^{-sl(\gamma)}. \end{aligned}$$

3. In case (b), for the super zeta function $Z^s(s; \sigma, \chi)$, we see by equation (2.22)

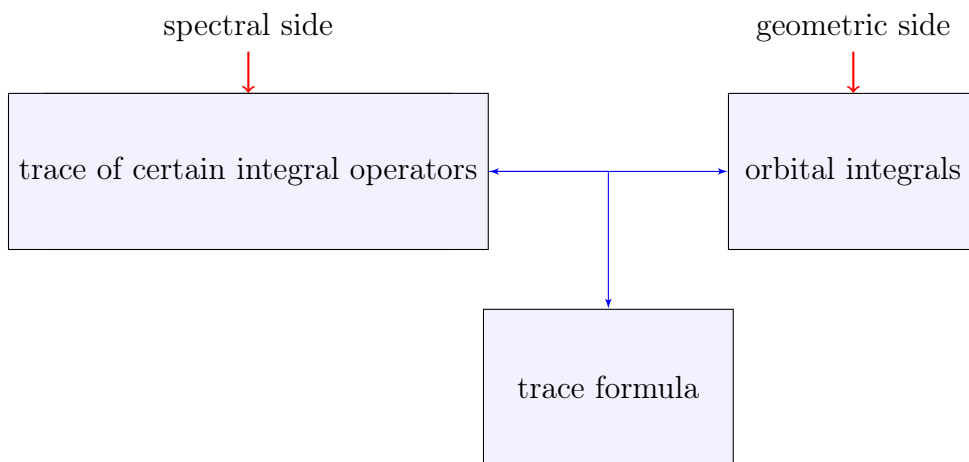
$$\begin{aligned} \frac{d}{ds} \log(Z^s(s; \sigma, \chi)) &= \frac{d}{ds} \log(Z(s; \sigma, \chi)) - \frac{d}{ds} \log(Z(s; w\sigma, \chi)) \\ &= \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} \operatorname{tr}(\chi(\gamma) \otimes \sigma(m_\gamma)) \frac{e^{-sl(\gamma)} e^{|\rho|l(\gamma)}}{\det(1 - \operatorname{Ad}(m_\gamma a_\gamma)_{\bar{\mathbb{N}}})} \\ &\quad - \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} \operatorname{tr}(\chi(\gamma) \otimes w\sigma(m_\gamma)) \frac{e^{-sl(\gamma)} e^{|\rho|l(\gamma)}}{\det(1 - \operatorname{Ad}(m_\gamma a_\gamma)_{\bar{\mathbb{N}}})} \\ &= \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} L_{sym}(\gamma; \sigma - w\sigma) e^{-sl(\gamma)}. \end{aligned}$$

□

CHAPTER 3

The Trace formula

The Selberg trace formula has turned out to be a useful tool for spectral analysis, which intersects with different areas of mathematics. The significance of this formula is that it combines spectral data, namely the eigenvalues of the Laplacian acting on smooth functions or on differential forms on a manifold, with geometric data such as the volume of the manifold, or the lengths of the closed geodesics.



The diagram above can be “translated” into the following trace formula (cf. Section 3.1):

$$\mathrm{Tr} R_{\Gamma}(h) = \sum_{[\gamma]} \mathrm{tr} \chi(\gamma) \mathrm{Vol}(\Gamma_{\gamma} \backslash G_{\gamma}) \int_{G_{\gamma} \backslash G} \mathrm{tr} h(g^{-1} \gamma g) dg,$$

where $h \in C_c^{\infty}(G)$ and $R_{\Gamma}(h)$ is an integral operator (cf. Definitions 3.1, 3.3 and Lemma 3.4).

3.1 The trace formula

We consider a connected semisimple Lie group G . Let Γ be a discrete cocompact subgroup of G . Let $\chi: \Gamma \rightarrow \text{GL}(V_\chi)$ be a finite dimensional unitary representation of Γ . Let $\|\cdot\|_\chi$ a norm induced by an hermitian inner product $\langle \cdot, \cdot \rangle$ on V_χ .

Let dg be an invariant Haar measure on G . By Proposition 1.21, there exists an invariant measure dx on $\Gamma \backslash G$, induced by the invariant measure on G . Let $L_\chi^2(\Gamma \backslash G)$ be the Hilbert space defined by

$$L_\chi^2(\Gamma \backslash G) := \{f: G \rightarrow V_\chi : f(\gamma g) = \chi(\gamma)f(g), \forall \gamma \in \Gamma, \forall g \in G, \int_{\Gamma \backslash G} \|f(g)\|_\chi dg < \infty\}.$$

Definition 3.1. We define the right regular representation R_Γ of G on the Hilbert space $L_\chi^2(\Gamma \backslash G)$ as the representation which associates every element $g_0 \in G$ with the operator $R_\Gamma(g_0)$ defined by

$$(R_\Gamma(g_0))f(g) := f(gg_0),$$

where $g \in G, f \in L_\chi^2(\Gamma \backslash G)$.

Remark 3.2. We can easily see that R_Γ is a unitary representation of G by the right invariance of the Haar measure on $\Gamma \backslash G$. Indeed, with respect to the L^2 -inner product (\cdot, \cdot) on $L_\chi^2(\Gamma \backslash G)$ we have for all $g, g_0 \in G$:

$$\begin{aligned} (R_\Gamma(g_0)f_1, R_\Gamma(g_0)f_2) &= \int_{\Gamma \backslash G} \langle f_1(gg_0), f_2(gg_0) \rangle dg \\ &= \int_{\Gamma \backslash G} \langle f_1(g), f_2(g) \rangle dg \\ &= (f_1, f_2). \end{aligned}$$

where we applied the change of variables $gg_0 \mapsto g$.

Definition 3.3. Let $h \in C_c^\infty(G)$. We define the operator $R_\Gamma(h)$ on $L_\chi^2(\Gamma \backslash G)$ by

$$R_\Gamma(h)f = \int_G h(g_0)R_\Gamma(g_0)f dg_0. \quad (3.1)$$

Lemma 3.4. The operator $R_\Gamma(h)$ is a linear bounded integral operator on $L_\chi^2(\Gamma \backslash G)$.

Proof. The linearity of $R_\Gamma(h)$ follows from the definition.

We let F be a fundamental domain for Γ . We compute

$$\begin{aligned}
(R_\Gamma(h)f)(g) &:= \int_G h(g_0)f(gg_0)dg_0 = \int_G h(g^{-1}g_0)f(g_0)dg_0 \\
&= \sum_{\gamma \in \Gamma} \int_{\gamma(F)} h(g^{-1}g_0)f(g_0)dg_0 \\
&= \sum_{\gamma \in \Gamma} \int_F h(g^{-1}\gamma g_0)f(\gamma g_0)dg_0 \\
&= \sum_{\gamma \in \Gamma} \int_F h(g^{-1}\gamma g_0)\chi(\gamma)f(g_0)dg_0 \\
&= \int_F \left(\sum_{\gamma \in \Gamma} h(g^{-1}\gamma g_0)\chi(\gamma) \right) f(g_0)dg_0.
\end{aligned}$$

Then, we see that the operator $R_\Gamma(h)f$ is an integral operator on $L^2_\chi(\Gamma \backslash G)$ and is given by

$$(R_\Gamma(h)f)(g) = \int_F K_h(g, g_0)f(g_0)dg_0,$$

with kernel

$$K_h(g, g_0) = \sum_{\gamma \in \Gamma} h(g^{-1}\gamma g_0)\chi(\gamma). \quad (3.2)$$

Since h is a compactly supported function on G , we may assume that $g^{-1}\gamma g_0$ belongs to a fixed compact set for all $\gamma \in \Gamma$. Also, $g, g_0 \in \overline{F}$, and so γ belongs to a compact set. Because the action of Γ on G is discrete, γ in (3.2) ranges only over a finite set.

Therefore, $K_h(g, g_0)$ is a smooth function on $G \times G$, and hence $R_\Gamma(h)$ is a bounded operator. \square

By [GGPS69, p.27], it follows that $R_\Gamma(h)$ is of trace class and its trace is

$$\mathrm{Tr} R_\Gamma(h) = \int_{\Gamma \backslash G} \mathrm{tr} K_h(g, g)dg = \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} \mathrm{tr} h(g^{-1}\gamma g) \mathrm{tr} \chi(\gamma)dg. \quad (3.3)$$

Denote by $\Gamma_\gamma := \mathrm{Centr}_\Gamma(\gamma)$ the centralizer of γ in Γ , and $G_\gamma := \mathrm{Centr}_G(\gamma)$ the centralizer of γ in G . The conjugacy classes of Γ are:

$$[\gamma] := \{\gamma' = \gamma_1\gamma(\gamma_1)^{-1} : \gamma_1 \in \Gamma\}.$$

We observe that the map

$$\gamma_1 \in \Gamma_\gamma \backslash \Gamma \mapsto \gamma_1\gamma\gamma_1^{-1} \in [\gamma]$$

is bijective. Hence, we have

$$\begin{aligned} \sum_{\gamma \in \Gamma} \int_F h(g^{-1}\gamma g) dg &= \sum_{[\gamma]} \sum_{\gamma_1 \in \Gamma_\gamma \backslash \Gamma} \int_F h(g^{-1}\gamma_1^{-1}\gamma\gamma_1 g) dg \\ &= \sum_{[\gamma]} \sum_{\gamma_1 \in \Gamma_\gamma \backslash \Gamma} \int_{\gamma_1(F)} h(g^{-1}\gamma g) dg \\ &= \sum_{[\gamma]} \int_{F_\gamma} h(g^{-1}\gamma g) dg, \end{aligned}$$

where $F_\gamma = \bigcup_{\gamma_1 \in \Gamma_\gamma \backslash \Gamma} \gamma_1 F$ is the fundamental domain of Γ_γ , i.e.

$$\int_{F_\gamma} h(g^{-1}\gamma g) dg = \sum_{\gamma_1 \in \Gamma_\gamma \backslash \Gamma} \int_F h(g^{-1}\gamma_1 g) dg. \quad (3.4)$$

We have

$$\begin{aligned} \int_{\Gamma_\gamma \backslash G} h(g^{-1}\gamma g) dg &= \int_{G_\gamma \backslash G} \int_{\Gamma_\gamma \backslash G_\gamma} h(g^{-1}g_1^{-1}\gamma g_1 g) dg_1 dg \\ &= \int_{G_\gamma \backslash G} \int_{\Gamma_\gamma \backslash G_\gamma} h(g^{-1}\gamma g) dg_1 dg \\ &= \text{Vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} h(g^{-1}\gamma g) dg. \end{aligned} \quad (3.5)$$

The space L^2_χ decomposes into an orthogonal direct sum of irreducible invariant subspaces (cf. [Wal76, p.171]). Let \widehat{G} be the set of equivalence classes of irreducible representations of G . Then,

$$R_\Gamma(h) = \sum_{\pi \in \widehat{G}} m_\Gamma(\pi) \pi(h),$$

where $m_\Gamma(\pi)$ is a non negative integer. Recall that $\pi(h)$ is a trace class operator. We set $\Theta_\pi(h) = \text{tr } \pi(h)$. Then,

$$\text{Tr } R_\Gamma(h) = \sum_{\pi \in \widehat{G}} m_\Gamma(\pi) \Theta_\pi(h),$$

Hence, if we combine equations (3.3), (3.4) and (3.5), we obtain the pre-trace formula

$$\sum_{\pi \in \widehat{G}} m_\Gamma(\pi) \Theta_\pi(h) = \sum_{[\gamma]} \text{tr } \chi(\gamma) \text{Vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} \text{tr } h(g^{-1}\gamma g) dg. \quad (3.6)$$

3.2 The trace formula for all locally symmetric spaces of real rank 1

We consider now the more special case of all locally symmetric spaces of real rank 1. Then, we can analyze the pre-trace formula (3.6) further and obtain a better version of it. Namely, we can explicitly write the right hand side of the formula in terms of the characters.

Definition 3.5. Let $P = MAN$ be the standard parabolic subgroup of G . For $(\sigma, V_\sigma) \in \widehat{M}$, we define the space $\mathcal{H}_\sigma^\infty$ of C^∞ -vectors in \mathcal{H}_σ as follows.

$$\mathcal{H}_\sigma^\infty := \left\{ f: G \rightarrow V_\sigma : f \in C^\infty, f(gman) = e^{-(i\lambda + |\rho|)t} \sigma^{-1}(m) f(g), \right. \\ \left. \forall g \in G, \forall man \in P \right\}, \quad (3.7)$$

where $\lambda \in \mathbb{C}$ and ρ as in (1.8), with norm

$$\|f\| = \int_K \|f(k)\|^2 dk. \quad (3.8)$$

We define the *principal series representation* as the induced representation

$$\pi_{\sigma, \lambda} := \text{Ind}_P^G(\sigma \otimes e^{i\lambda} \otimes \text{Id}) \quad (3.9)$$

with representation space the Hilbert space \mathcal{H}_σ , obtained by completion of $\mathcal{H}_\sigma^\infty$ with respect to the norm in (3.8). For $f \in \mathcal{H}_\sigma$, the action of G on f is given by

$$\pi_{\sigma, \lambda}(g)f(g') = f(g^{-1}g').$$

Remark 3.6. 1. $\mathfrak{a}_\mathbb{C}^*$ is the space of the linear functional on $\mathfrak{a}_\mathbb{C}$. In (3.7), λ is a complex number. Hence, $\mathfrak{a}_\mathbb{C}^*$ is identified with \mathbb{C} , using the positive root. If $\lambda \in \mathbb{R}$, then the representation $\pi_{\sigma, \lambda}$ is unitary. This is proved in [Kna86, Chapter VII, §2]. Our definition is slightly different because we use the exponential of $i\lambda \in \mathbb{C}$ instead of $\lambda \in \mathbb{C}$, but the two definitions are equivalent.

2. In addition, if $\lambda \in \mathbb{R} - \{0\}$, then $\pi_{\sigma, \lambda}$ is irreducible.

3. Let w be a non-trivial element in W_A . Then for $\sigma' = w\sigma \in \widehat{M}$ and $\lambda' = -\lambda \in \mathbb{C}$, $\pi_{\sigma', \lambda'}$ and $\pi_{\sigma, \lambda}$ are equivalent.

4. If we restrict the principal series representation $\pi_{\sigma, \lambda}$ of G to K , then it follows by the Frobenius Reciprocity Principle ([Kna86, p.208]) that for $\tau \in \widehat{K}$ and $\sigma \in \widehat{M}$,

$$[\pi_{\sigma, \lambda} : \tau] = [\tau|_M : \sigma]. \quad (3.10)$$

For $\lambda \in \mathbb{R}$ and $h \in C_c^\infty(G)$, let $\pi_{\sigma,\lambda}(h)$ be the linear operator on \mathcal{H}_σ defined by

$$\pi_{\sigma,\lambda}(h) := \int_G h(g)\pi_{\sigma,\lambda}(g)dg. \quad (3.11)$$

Then by [War72, Theorem 4.5.7.4], it is a well defined trace class operator. We define the character of $\pi_{\sigma,\lambda}$, by

$$\Theta_{\sigma,\lambda}(h) := \text{Tr}(\pi_{\sigma,\lambda}(h)), \quad h \in C_c^\infty(G). \quad (3.12)$$

We recall here the definition of the Harish-Chandra L^q -Schwartz space $\mathcal{C}^q(G) \otimes V_\tau$. Following [BM83, p.161-162], we define the function

$$D(g) := d(gK, eK), \quad g \in G,$$

where $d(g_1K, g_2K)$ denotes the geodesic distance between two elements g_1K, g_2K in G/K . We consider the unitary representation

$$\pi_0 := \text{Ind}_P^G(\text{Id} \otimes e^{i\lambda} \otimes \text{Id}),$$

i.e. the unitary representation induced by the trivial representation of M with representation space \mathcal{H}_0 . Let $\pi_0|_K$ be the restriction of π_0 to K . Let $\xi \in \mathcal{H}_0$ be the unit vector such that

$$\pi_0(k)\xi = \xi, \quad \forall k \in K.$$

Then, we define the matrix-coefficient functional

$$\Xi(g) := (\pi_0(g)\xi, \xi),$$

where (\cdot, \cdot) denotes the corresponding inner product in \mathcal{H}_0 . We define

$$\mathcal{C}^q(G) := \{f \in C^\infty(G) : \sup_{g \in G} ((1+D(g))^m \Xi(g)^{-2/q} |L(X)R(X')f(g)|) < \infty, \forall m \in \mathbb{R}^+\},$$

where $X, X' \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$, and $L(X), R(X)$ denote the left and respectively the right invariant differential operator on G associated with an element $X \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$. We turn now to the equation (3.6).

Since $\text{rank}_\mathbb{R} G = 1$, we know by Lemma 2.1 that every element $\gamma \in \Gamma - \{e\}$ is conjugate in G to an element $m_\gamma a_\gamma \in MA^+$. Moreover, since $\text{rank}_\mathbb{R} G = 1$, the centralizer Γ_γ of γ in Γ is infinite cyclic. Thus, there exists a $\gamma_0 \in \Gamma_\gamma$ that generates Γ_γ and $\gamma = \gamma_0^{n(\gamma)}$, for some integer $n(\gamma) > 1$.

Since G_γ/A is compact, the last line in equation (3.5) gives

$$\text{Vol}(G_\gamma/A) \int_{G_\gamma \setminus G} h(g^{-1}\gamma g)dg = \int_{G/A} h(g^{-1}\gamma g)d\dot{g}, \quad (3.13)$$

where the measure $d\dot{g}$ is defined by

$$\int_G h(g)dg = \int_{G/A} \int_{\mathbb{R}} h(g \exp(tH_{\mathbb{R}})) dt d\dot{g}.$$

Therefore, equation (3.6) becomes

$$\mathrm{Tr} R_{\Gamma}(h) = \sum_{[\gamma]} \mathrm{tr} \chi(\gamma) \frac{\mathrm{Vol}(\Gamma_{\gamma} \backslash G_{\gamma})}{\mathrm{Vol}(G_{\gamma}/A)} \int_{G/A} \mathrm{tr} h(g^{-1}\gamma g) d\dot{g}, \quad (3.14)$$

Now by the observations above,

$$\frac{\mathrm{Vol}(\Gamma_{\gamma} \backslash G_{\gamma})}{\mathrm{Vol}(G_{\gamma}/A)} = l(\gamma_0).$$

Hence, equation (3.14) becomes

$$\mathrm{Tr} R_{\Gamma}(h) = \sum_{[\gamma]} \mathrm{tr} \chi(\gamma) l(\gamma_0) \int_{G/A} \mathrm{tr} h(g^{-1}\gamma g) d\dot{g}.$$

If we separate the conjugacy class of the identity element e we get

$$\mathrm{Tr} R_{\Gamma}(h) = \dim(V_{\chi}) \mathrm{Vol}(X) h(e) + \sum_{[\gamma] \neq e} \mathrm{tr} \chi(\gamma) l(\gamma_0) \int_{G/A} \mathrm{tr} h(g^{-1}\gamma g) d\dot{g}. \quad (3.15)$$

In the right hand side of (3.15) the first term gives the identity contribution, while the second one the hyperbolic.

By [Wal73, Corollaries 7.7.10 and 7.7.11],

$$\int_{G/A} h(g^{-1}\gamma g) d\dot{g} = \frac{1}{D(m_{\gamma}a_{\gamma})} \mathcal{F}_h(m_{\gamma}a_{\gamma}), \quad (3.16)$$

where $\mathcal{F}_h(m_{\gamma}a_{\gamma}) = \mathcal{A}(h)$ is the Abel transform of h given by

$$\mathcal{A}(h)(a_{\gamma}) = \mathcal{F}_h(m_{\gamma}a_{\gamma}) := e^{|\rho|l(\gamma)} \int_K \int_N h(km_{\gamma}a_{\gamma}nk^{-1}) dk dn, \quad (3.17)$$

and $D(m_{\gamma}a_{\gamma})$ is defined to be

$$D(m_{\gamma}a_{\gamma}) := e^{|\rho|l(\gamma)} \det(\mathrm{Id} - \mathrm{Ad}(m_{\gamma}a_{\gamma})|_{\bar{\mathfrak{n}}}).$$

Hence, equation (3.15) gives

$$\mathrm{Tr} R_{\Gamma}(h) = \dim(V_{\chi}) \mathrm{Vol}(X) h(e) + \sum_{[\gamma] \neq e} \mathrm{tr} \chi(\gamma) l(\gamma_0) \frac{1}{D(m_{\gamma}a_{\gamma})} \mathcal{F}_h(m_{\gamma}a_{\gamma}). \quad (3.18)$$

By [Wal73, Theorem 8.8.2],

$$\Theta_{\sigma,\lambda}(h) = \operatorname{tr} \sigma(m_\gamma) \int_{\mathbb{R}} \int_M \mathcal{F}_h(m_\gamma a_\gamma) e^{i\lambda t} dm dt.$$

By this representation of $\Theta_{\sigma,\lambda}(h)$, we get that $\mathcal{F}(m_\gamma a_\gamma)$ is just the Fourier transform of the character $\Theta_{\sigma,\lambda}(h)$. Hence,

$$\mathcal{F}_h(m_\gamma a_\gamma) = \frac{1}{2\pi} \sum_{\sigma \in \widehat{M}} \overline{\operatorname{tr}(\sigma(m_\gamma))} \int_{\mathbb{R}} \Theta_{\sigma,\lambda}(h) e^{-i\lambda t} d\lambda. \quad (3.19)$$

We substitute (3.19) in (3.18) and get

$$\operatorname{Tr} R_\Gamma(h) = \dim(V_\chi) \operatorname{Vol}(X) h(e) + \sum_{[\gamma] \neq e} \sum_{\sigma \in \widehat{M}} \frac{\overline{\operatorname{tr}(\sigma(m_\gamma))} \operatorname{tr} \chi(\gamma) l(\gamma_0)}{2\pi D(m_\gamma a_\gamma)} \int_{\mathbb{R}} \Theta_{\sigma,\lambda}(h) e^{-i\lambda l(\gamma)} d\lambda. \quad (3.20)$$

We specialize now to our case, where $G = \operatorname{Spin}(d, 1)$, $K = \operatorname{Spin}(d)$, and Γ is a discrete torsion-free cocompact subgroup of G . Since $\operatorname{rank}(G) > \operatorname{rank}(K)$, by a classical result of Harish-Chandra ([HC66]), the set of the discrete series representations of G is empty. Then, formula (3.20) can be developed further. In particular, we can evaluate the term $h(e)$ as the *inverse Fourier transform* of h at e . By (3.11) we write the inverse Fourier transform of h at $g \in G$ as

$$h(g) = \int \operatorname{Tr}(\pi_{\sigma,\lambda}(h) \pi_{\sigma,\lambda}(g)^*) d\mu_{PL}(\pi_{\sigma,\lambda}),$$

where $d\mu_{PL}(\pi_{\sigma,\lambda})$ is the the Plancherel measure, viewed as a measure on the set of the principal series representations $\pi_{\sigma,\lambda}$. It is sufficient to compute

$$h(e) = \int \Theta_{\sigma,\lambda}(h) d\mu_{PL}(\pi_{\sigma,\lambda}). \quad (3.21)$$

By [Kna86, Theorem 13.2],

$$h(e) = \sum_{\sigma \in \widehat{M}} \int_{\mathbb{R}} \Theta_{\sigma,\lambda}(h) d\mu_{PL}(\pi_{\sigma,\lambda}),$$

where

$$d\mu_{PL}(\pi_{\sigma,\lambda}) = P_\sigma(i\lambda) d\lambda. \quad (3.22)$$

Here, $P_\sigma(i\lambda)$ is the Plancherel polynomial given by

$$P_\sigma(i\lambda) = \prod_{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})} \frac{\langle i\lambda + \nu_\sigma + \rho_{\mathfrak{m}}, \alpha \rangle}{\langle \rho_{\mathfrak{g}}, \alpha \rangle},$$

where $\langle \cdot, \cdot \rangle$ is the inner product as in (1.2), ν_σ is the highest weight of σ as in (1.10), $\rho_{\mathfrak{m}}$ is defined as in (1.9), and $\rho_{\mathfrak{g}}$ is defined by

$$\rho_{\mathfrak{g}} := \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})} \alpha.$$

Let $z = i\lambda \in \mathbb{C}$. Then, by [Mia79, p.264-265], $P_\sigma(z)$ is an even polynomial of z . We have

$$P_\sigma(z) = P_\sigma(-z) = P_{w\sigma}(z).$$

CHAPTER 4

The twisted Bochner-Laplace operator

4.1 Non-unitary representations of Γ . General setting

Let $E \rightarrow X$ be a complex vector bundle equipped with a connection ∇ . We define the second covariant derivative ∇^2 by

$$\nabla_{V,W}^2 := \nabla_V \nabla_W - \nabla_{\nabla_V^{LC} W},$$

where ∇^{LC} denotes the Levi-Civita connection on the tangent bundle of X and $V, W \in C^\infty(X, TX)$. We note here that at any point $x \in X$, the operator $\nabla_{V,W}^2$ depends only on V_x and W_x . The dependence of the second derivative on V_x follows directly from the properties of the covariant derivative ∇_V . For the dependence on W_x , one can use the identity

$$\nabla_{V,W}^2 - \nabla_{W,V}^2 = R_{V,W},$$

where $R_{\cdot,\cdot}$ is the curvature tensor of E .

We define now the connection Laplacian to be the negative of the trace of the second covariant derivative ∇^2 :

$$\Delta_E := -\text{Tr } \nabla^2. \tag{4.1}$$

By [LM89, p.154], the connection Laplacian equals the Bochner-Laplace operator, i.e.,

$$\Delta_E = \nabla^* \nabla.$$

In terms of a local orthonormal frame field (e_1, \dots, e_d) of $T_x X$, for $x \in X$, the connection Laplacian is given by

$$\Delta_E = - \sum_{j=1}^d \nabla_{e_j, e_j}^2.$$

Then, $\Delta_E : C^\infty(X, E) \rightarrow C^\infty(X, E)$ is a second-order differential operator.

We want to compute the principal symbol of the connection Laplacian. Let U be an open subset of X . Let (x_1, \dots, x_d) be local coordinates on X at $x \in X$ and (s_1, \dots, s_k) be a local frame field of E over U . Here, $k = \text{rank}(E)$. Then, every smooth section $\phi \in C^\infty(U, E|_U)$ can be written as

$$\phi = \sum_{a=1}^k \phi^a s_a.$$

Let $Y \in C^\infty(X, TX)$. Locally, it can be expressed as $Y = \sum_{j=1}^d Y_j \frac{\partial}{\partial x_j}$. The covariant derivative $\nabla_Y \sigma$ locally can be described as

$$\nabla_Y \phi = \sum_{j=1}^d \sum_{a=1}^k Y_j \left(\frac{\partial}{\partial x_j} \phi^a + \sum_{\beta=1}^k \omega_j^{\alpha, \beta} \phi^\beta \right) s_a,$$

where $\omega_j^{\alpha, \beta}$ denotes a $k \times k$ matrix of 1-forms on U with values in $\text{End}(E)$. Then, we associate every local tangent vector e_j with $\partial/\partial x_j$. Also, under the identification $T_x X \cong T_x^* X$, we correspond every e_j to dx_j . Let $\xi = \sum_{i=1}^d \xi_i dx_i$ be a cotangent vector in $T_x^* X$ with $\xi \neq 0$. By the local formula for the covariant derivative $\nabla_X \sigma$ and the definition of the connection Laplacian, we have

$$\Delta_E = - \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2} + \text{lower order terms.}$$

Hence,

$$\sigma_{\Delta_E}(x, \xi) = -(i)^2 \sum_{j=1}^d \xi_j^2 \text{Id}_{E_x} = \|\xi\|_x^2 \text{Id}_{E_x}. \quad (4.2)$$

Let h be a metric in E . Then, Δ_E acts in $L^2(X, E)$ with domain $C^\infty(X, E)$. By (4.2), we can conclude that Δ_E is an elliptic operator and hence it has nice spectral properties. Namely, its spectrum is discrete and contained in a translate of a positive cone $C \subset \mathbb{C}$ such that $\mathbb{R}^+ \subset C$ (Appendix A, Lemma A.8). Furthermore, if the metric is compatible with the connection ∇ , Δ_E is formally self adjoint.

Let $\chi: \Gamma \rightarrow \mathrm{GL}(V_\chi)$ be an arbitrary representation $\chi: \Gamma \rightarrow \mathrm{GL}(V_\chi)$ of Γ . Let $E_\chi \rightarrow X$ be the associated flat vector bundle over X , equipped with a flat connection ∇^{E_χ} .

We specialize now to the twisted case $E = E_0 \otimes E_\chi$, where $E_0 \rightarrow X$ is a complex vector bundle equipped with a connection ∇^{E_0} and a metric compatible with this connection.

Let $\nabla^E = \nabla^{E_0 \otimes E_\chi}$ the product-connection, which is defined by

$$\nabla^{E_0 \otimes E_\chi} := \nabla^{E_0} \otimes 1 + 1 \otimes \nabla^{E_\chi}.$$

We define the operator $\Delta_{E_0, \chi}^\sharp$ by the same formula

$$\Delta_{E_0, \chi}^\sharp = -\mathrm{Tr}((\nabla^{E_0 \otimes E_\chi})^2). \quad (4.3)$$

We choose a hermitian metric in E_χ . Then, $\Delta_{E_0, \chi}^\sharp$ acts on $L^2(X, E_0 \otimes E_\chi)$. However, it is not a formally self-adjoint operator in general.

We want to describe this operator locally. Following the analysis in [Mül11], we consider an open subset U of X such that $E_\chi|_U$ is trivial. Then, $E_0 \otimes E_\chi|_U$ is isomorphic to the direct sum of m -copies of $E_0|_U$

$$(E_0 \otimes E_\chi)|_U \cong \bigoplus_{i=1}^m E_0|_U,$$

where $m := \mathrm{rank}(E_\chi) = \dim V_\chi$.

Let $(e_i), i = 1, \dots, m$, be a basis of flat sections of $E_\chi|_U$. Then, each $\phi \in C^\infty(U, (E_0 \otimes E_\chi)|_U)$ can be written as

$$\phi = \sum_{i=1}^m \phi_i \otimes e_i,$$

where $\phi_i \in C^\infty(U, E_0|_U), i = 1, \dots, m$.

The product connection is given by

$$\nabla_Y^{E_0 \otimes E_\chi} \phi = \sum_{i=1}^m (\nabla_Y^{E_0})(\phi_i) \otimes e_i,$$

where $Y \in C^\infty(X, TX)$.

By (4.3) we obtain the twisted Bochner-Laplace operator

$$\Delta_{E_0, \chi}^\sharp \phi = \sum_{i=1}^m (\Delta_{E_0} \phi_i) \otimes e_i, \quad (4.4)$$

where Δ_{E_0} denotes the Bochner-Laplace operator $\Delta^{E_0} = (\nabla^{E_0})^* \nabla^{E_0}$ associated with the connection ∇^{E_0} .

Let now $\tilde{E}_0, \tilde{E}_\chi$ be the pullbacks to \tilde{X} of E_0, E_χ , respectively. Then,

$$\tilde{E}_\chi \cong \tilde{X} \times V_\chi.$$

We have

$$C(\tilde{X}, \tilde{E}_0 \otimes \tilde{E}_\chi) \cong C(\tilde{X}, \tilde{E}_0) \otimes V_\chi. \quad (4.5)$$

With respect to the isomorphism (4.5), it follows from (4.4) that the lift of $\Delta_{E_0, \chi}^\sharp$ to \tilde{X} takes the form

$$\tilde{\Delta}_{E_0, \chi}^\sharp = \tilde{\Delta}_{E_0} \otimes \text{Id}_{V_\chi}, \quad (4.6)$$

where $\tilde{\Delta}_{E_0}$ is the lift of Δ_{E_0} to \tilde{X} .

Remark 4.1. *The description of the twisted Bochner-Laplace operator is not different from the general idea of the flat Laplacian. Indeed, it is a special case of the flat Laplacian acting on the space of E_χ -valued p -forms. Recall the Hodge-Laplace operator $\Delta_\chi = d_\chi \delta_\chi + \delta_\chi d_\chi: \Lambda^p(X, E_\chi) \rightarrow \Lambda^p(X, E_\chi)$, defined as in [MM63, Section 2], where $d_\chi: \Lambda^p(X, E_\chi) \rightarrow \Lambda^{p+1}(X, E_\chi)$ denotes the exterior derivative and δ_χ is the formal adjoint of d_χ with respect to inner product in $\Lambda^p(X, E_\chi)$. Then, the flat Laplacian Δ_χ^\sharp is defined analogously (cf. [Fay81], and [BK05],[BK08]). The difference is that we define a new $*$ -operator*

$$*_\chi := * \otimes \text{Id}_{V_\chi}$$

acting on $C^\infty(X, \Lambda^p T^* X \otimes E_\chi)$, where $*$ is the usual Hodge $*$ -operator acting on $C^\infty(X, \Lambda^p T^* X)$. We define the operator δ_χ^\sharp by

$$\delta_\chi^\sharp := (-1)^{d(p+1)+1} *_\chi d_\chi *_\chi.$$

We define the flat Hodge-Laplace operator or the twisted Laplace operator Δ_χ^\sharp acting on $\Lambda^p(X, E_\chi)$ by

$$\Delta_\chi^\sharp := \delta_\chi^\sharp d_\chi + d_\chi \delta_\chi^\sharp.$$

This operator is not self-adjoint with respect to the inner product in $C^\infty(X, \Lambda^p T^* X \otimes E_\chi)$ induced by any hermitian metric h^χ in E_χ and the riemannian metric in TX . For a detailed definition and analytic properties of the flat Laplacian on p -forms see Chapter 9.

The twisted Laplace operator fits into the framework described above by the the Weizenböck formula ([LM89, Corollary 8.3]), which connects the Hodge-Laplace operator and the connection Laplacian up to the Ricci transformation. Namely,

$$\Delta_\chi^\sharp = \nabla^* \nabla + \text{Ric}.$$

Here Ric denotes the Ricci transformation of TX . For $\phi \in TX$, it is defined by

$$Ric(\phi) := \sum_{i=1}^d R_{e_j, \phi}(e_j),$$

where (e_1, \dots, e_d) is a local orthonormal frame field of $T_x X$, $x \in X$, and $R_{\cdot, \cdot}$ denotes the riemannian curvature transformation of TX .

We turn to the case of the twisted Bochner-Laplace operator. As already mentioned, this operator is not self-adjoint with respect to the inner product in $C^\infty(X, E_0 \otimes E_\chi)$ induced by the riemannian metric and the tensor product of the metrics in E_χ and E_0 . Nevertheless, by (4.4) $\Delta_{E_0, \chi}^\sharp$ has principal symbol

$$\sigma_{\Delta_{E_0, \chi}^\sharp}(x, \xi) = \|\xi\|_x^2 \text{Id}_{(E_0 \otimes E_\chi)_x}, \quad x \in X, \xi \in T_x^* X, \xi \neq 0.$$

Hence, it has nice spectral properties, i.e. its spectrum is discrete and contained in a translate of a positive cone $C \subset \mathbb{C}$ such that $\mathbb{R}^+ \subset C$ (Appendix A, Lemma A.8).

In Section 4.2, we consider the corresponding heat semi-group $e^{-t\Delta_{E_0, \chi}^\sharp}$. We can apply the Lidskii's theorem, which gives a general expression for the trace of a trace class, (not necessarily self-adjoint) operator in terms of its eigenvalues λ_j and the corresponding algebraic multiplicities $m(\lambda_j)$ (cf. Appendix A). Namely, by [Sim05, Theorem 3.7],

$$\text{Tr} e^{-t\Delta_{E_0, \chi}^\sharp} = \sum_{\lambda_j \in \text{spec}(\Delta_{E_0, \chi}^\sharp)} m(\lambda_j) e^{-t\lambda_j}. \quad (4.7)$$

4.2 The heat kernel on the universal covering

We study the Bochner-Laplace operator $\tilde{\Delta}_\tau$, associated with a complex finite dimensional unitary representation (τ, V_τ) of K , on the universal covering \tilde{X} . The associated heat operator $e^{-t\tilde{\Delta}_\tau}$ is an integral operator with smooth kernel (cf. Appendix B). Our goal is to write explicitly the corresponding trace formula for the heat operator associated to the twisted Bochner-Laplace operator $\Delta_{\tau, \chi}^\sharp$. We start with the following definitions.

We regard the Lie group G as principal K -fiber bundle over \tilde{X} . Let $\pi: G \rightarrow G/K$ be the canonical projection. Then, since \mathfrak{p} is invariant under the adjoint action $\text{Ad}(k)$, $k \in K$, the assignment

$$T_g^{hor} := \left. \frac{d}{dt} \right|_{t=0} g \exp(tX), \quad X \in \mathfrak{p}$$

defines a horizontal distribution on G ([KN96, Chapter III]). This is the canonical connection in the principal bundle G .

Let $\tau : K \rightarrow \text{GL}(V_\tau)$ be a complex finite dimensional unitary representation of K on a vector space V_τ , equipped with an inner product $\langle \cdot, \cdot \rangle_\tau$. Let \tilde{E}_τ be the homogenous vector bundle associated to (τ, V_τ) , defined by

$$\tilde{E}_\tau := G \times_\tau V_\tau \rightarrow \tilde{X},$$

where K acts on (G, V_τ) on the right by

$$(g, v)k = (gk, \tau^{-1}(k)v), \quad g \in G, k \in K, v \in V_\tau.$$

The inner product $\langle \cdot, \cdot \rangle_\tau$ on the vector space V_τ induces a G -invariant metric h^{E_τ} on \tilde{E}_τ .

We denote by $C^\infty(\tilde{X}, \tilde{E}_\tau)$ the space of the smooth sections of the vector bundle \tilde{E}_τ .

We define the space

$$C^\infty(G; \tau) = \{f : G \rightarrow V_\tau : f \in C^\infty, f(gk) = \tau(k)^{-1}f(g), \forall g \in G, \forall k \in K\}. \quad (4.8)$$

Similarly, we denote by $C_c^\infty(G; \tau)$ the subspace of $C^\infty(G; \tau)$ of compactly supported functions, and by $L^2(G; \tau)$ the completion of $C_c^\infty(G; \tau)$ with respect to the inner product

$$\langle f, h \rangle = \int_{G/K} \langle f(g), h(g) \rangle_\tau dg.$$

Let $A : C^\infty(\tilde{X}, \tilde{E}_\tau) \rightarrow C^\infty(G; \tau)$ be the operator, defined by

$$Af(g) = g^{-1}f(gK).$$

Then the canonical connection on \tilde{E}_τ is given by

$$\begin{aligned} A(\nabla_{d\pi(g)X}^\tau f)(g) &= \left. \frac{d}{dt} \right|_{t=0} Af(g \exp(tX)) \\ &= \left. \frac{d}{dt} \right|_{t=0} (g \exp(tX))^{-1} f(g \exp(tX)K), \end{aligned}$$

where $g \in G, X \in \mathfrak{p}$, and $f \in C^\infty(\tilde{X}, \tilde{E}_\tau)$. By [Mia80, p.4], A induces a canonical isomorphism

$$C^\infty(\tilde{X}, \tilde{E}_\tau) \cong C^\infty(G; \tau). \quad (4.9)$$

Similarly, there exist the following isomorphisms

$$\begin{aligned} C_c^\infty(\tilde{X}, \tilde{E}_\tau) &\cong C_c^\infty(G; \tau) \\ L^2(\tilde{X}, \tilde{E}_\tau) &\cong L^2(G; \tau). \end{aligned} \quad (4.10)$$

We consider the Bochner-Laplace operator associated with $\tilde{\nabla}^\tau$,

$$\tilde{\Delta}_\tau = (\tilde{\nabla}^\tau)^* \tilde{\nabla}^\tau : C_c^\infty(\tilde{X}, \tilde{E}_\tau) \rightarrow L^2(\tilde{X}, \tilde{E}_\tau).$$

Let now $\Omega \in Z(\mathfrak{g}_\mathbb{C})$ be the Casimir element as it is defined in Section 1.1 (Definition 1.7). We assume that τ is irreducible. Let $\Omega|_K \in Z(\mathfrak{k})$ be the Casimir element of K and λ_τ the associated Casimir eigenvalue (cf. Section 1.1). Then, with respect to the isomorphism (4.10), the Bochner-Laplace operator acting on $C_c^\infty(G; \tau)$ is given by

$$\tilde{\Delta}_\tau = -R(\Omega) + \lambda_\tau \text{Id}. \quad (4.11)$$

This is proved in [Mia80, Proposition 1.1].

The operator $\tilde{\Delta}_\tau$ is an elliptic formally self-adjoint differential operator of second order. By [Che73], it is an essentially self-adjoint operator. Its self-adjoint extension will be also denoted by $\tilde{\Delta}_\tau$.

We consider the corresponding heat semi-group $e^{-t\tilde{\Delta}_\tau}$ acting on the space $L^2(\tilde{X}, \tilde{E}_\tau)$.

$$e^{-t\tilde{\Delta}_\tau} : L^2(\tilde{X}, \tilde{E}_\tau) \rightarrow L^2(\tilde{X}, \tilde{E}_\tau).$$

By [CY81, p.467], $e^{-t\tilde{\Delta}_\tau}, t > 0$ is an infinitely smoothing operator with a C^∞ -kernel, i.e. there exists a smooth function $k_t^\tau : G \times G \rightarrow \text{End}(V_\tau)$ such that

1. it is symmetric in the G -variables and for each $g \in G$, the map $g' \mapsto k_t^\tau(g, g')$ belongs to $L^2(\tilde{X}, \tilde{E}_\tau)$;
2. it satisfies the covariance property,

$$k_t^\tau(gk, g'k') = \tau^{-1}(k)k_t^\tau(g, g')\tau(k'), \quad \forall g, g' \in G, k, k' \in K;$$

3. for $f \in L^2(\tilde{X}, \tilde{E}_\tau)$:

$$e^{-t\tilde{\Delta}_\tau} f(g) = \int_G k_t^\tau(g, g') f(g') dg'. \quad (4.12)$$

The Casimir element is invariant under the action of G . Hence, $\tilde{\Delta}_\tau$ is G -invariant, and $e^{-t\tilde{\Delta}_\tau}$ is an integral operator which commutes with the right regular representation of G in $L^2(\tilde{X}, \tilde{E}_\tau)$. Then there exists a function $H_t^\tau : G \rightarrow \text{End}(V_\tau)$, such that

1. $H_t^\tau(g^{-1}g') = k_t^\tau(g, g'), \quad \forall g, g' \in G$;

2. it satisfies the covariance property

$$H_t^\tau(kgk') = \tau^{-1}(k)H_t^\tau(g)\tau(k'), \quad \forall g \in G, \forall k, k' \in K; \quad (4.13)$$

3. for $f \in L^2(\tilde{X}, \tilde{E}_\tau)$:

$$e^{-t\tilde{\Delta}_\tau} f(g) = \int_G H_t^\tau(g^{-1}g') f(g') dg'. \quad (4.14)$$

We consider the space $(\mathcal{C}^q(G) \otimes \text{End}(V_\tau))^{K \times K}$ the Harish-Chandra L^q -Schwartz space of functions on G with values in $\text{End}(V_\tau)$ such that the covariance property (4.13) is satisfied.

Theorem 4.2. *Let $t > 0$. Then, for every $q > 0$*

$$H_t^\tau \in (\mathcal{C}^q(G) \otimes \text{End}(V_\tau))^{K \times K}.$$

Proof. This is proved in [BM83, Proposition 2.4]. □

In [BM83, p.161] it is proved that

$$e^{-t\tilde{\Delta}_\tau} = R_\Gamma(H_t^\tau),$$

where $R_\Gamma(H_t^\tau)$ denotes the bounded trace class operator, induced by the right regular representation of G (Definition 3.1), acting on $C^\infty(G; \tau)$. It is described by the formula

$$e^{-t\tilde{\Delta}_\tau} f(g) = \int_G H_t^\tau(g^{-1}g') f(g') dg'.$$

More generally, we consider a unitary admissible representation π of G in a Hilbert space \mathcal{H}_π . We set

$$\tilde{\pi}(H_t^\tau) = \int_G \pi(g) \otimes H_t^\tau(g) dg.$$

This defines a bounded trace class operator on $\mathcal{H}_\pi \otimes V_\tau$. By [BM83, p.160-161], relative to the splitting

$$\mathcal{H}_\pi \otimes V_\tau = (\mathcal{H}_\pi \otimes V_\tau)^K \oplus [(\mathcal{H}_\pi \otimes V_\tau)^K]^\perp,$$

$\tilde{\pi}(H_t^\tau)$ has the form

$$\tilde{\pi}(H_t^\tau) = \begin{pmatrix} \pi(H_t^\tau) & 0 \\ 0 & 0 \end{pmatrix}, \quad (4.15)$$

with $\pi(H_t^\tau)$ acting on $(\mathcal{H}_\pi \otimes V_\tau)^K$. Then, it follows that

$$e^{-t(-\pi(\Omega) + \lambda_\tau)} \text{Id} = \pi(H_t^\tau), \quad (4.16)$$

where Id denotes the identity on the space $(\mathcal{H}_\pi \otimes V_\tau)^K$ ([BM83, Corollary 2.2]). We let

$$h_t^\tau(g) := \text{tr } H_t^\tau(g).$$

We consider orthonormal bases $(\xi_n), n \in \mathbb{N}, (e_j), j = 1, \dots, k$ of the vector spaces \mathcal{H}_π, V_τ , respectively, where $k := \dim(V_\tau)$. By (4.15),

$$\mathrm{Tr}(\pi(H_t^\tau)) = \mathrm{Tr}(\tilde{\pi}(H_t^\tau)).$$

We have

$$\begin{aligned} \mathrm{Tr}(\tilde{\pi}(H_t^\tau)) &= \sum_n \sum_j \langle \tilde{\pi}(H_t^\tau)(\xi_n \otimes e_j), (\xi_n \otimes e_j) \rangle \\ &= \sum_n \sum_j \int_G \langle \pi(g)\xi_n, \xi_n \rangle \langle H_t^\tau(g)e_j, e_j \rangle dg \\ &= \sum_n \int_G \langle \pi(g)\xi_n, \xi_n \rangle h_t^\tau(g) dg \\ &= \sum_n \langle \pi(h_t^\tau)\xi_n, \xi_n \rangle \\ &= \mathrm{Tr} \pi(h_t^\tau). \end{aligned} \tag{4.17}$$

Hence, if we combine equations (4.16) and (4.17), we have

$$\mathrm{Tr} \pi(h_t^\tau) = e^{-t(-\pi(\Omega)+\lambda_\tau)} \dim(\mathcal{H}_\pi \otimes V_\tau)^K. \tag{4.18}$$

Now we want to specify the unitary representation π of G . We consider the unitary principal series representation $\pi_{\sigma,\lambda}$, defined in Section 3.2. Our goal is to compute the *Fourier transform* of h_t^τ ,

$$\Theta_{\sigma,\lambda}(h_t^\tau) = \mathrm{Tr} \pi_{\sigma,\lambda}(h_t^\tau)$$

(cf. Section 3.2).

Proposition 4.3. *For $\sigma \in \widehat{M}$ and $\lambda \in \mathbb{R}$ let $\Theta_{\sigma,\lambda}$ be the global character of $\pi_{\sigma,\lambda}$. Let $\tau \in \widehat{K}$. Then,*

$$\Theta_{\sigma,\lambda}(h_t^\tau) = e^{-t(-\pi_{\sigma,\lambda}(\Omega)+\lambda_\tau)}. \tag{4.19}$$

Proof. We have

$$\Theta_{\sigma,\lambda}(h_t^\tau) = e^{-t(-\pi_{\sigma,\lambda}(\Omega)+\lambda_\tau)} \dim(\mathcal{H}_{\pi_{\sigma,\lambda}} \otimes V_\tau)^K = e^{-t(-\pi_{\sigma,\lambda}(\Omega)+\lambda_\tau)} [\pi_{\sigma,\lambda} : \check{\tau}],$$

where $\check{\tau}$ denotes the contragredient representation of τ . If we use the fact that $\check{\tau} \cong \tau$ ([Oni04, Proposition 4 in §4 and Proposition 3 in §7]), we obtain

$$\begin{aligned} \Theta_{\sigma,\lambda}(h_t^\tau) &= e^{-t(-\pi_{\sigma,\lambda}(\Omega)+\lambda_\tau)} [\pi_{\sigma,\lambda} : \check{\tau}] \\ &= e^{-t(-\pi_{\sigma,\lambda}(\Omega)+\lambda_\tau)} [\pi_{\sigma,\lambda} : \tau] = e^{-t(-\pi_{\sigma,\lambda}(\Omega)+\lambda_\tau)} [\tau \upharpoonright_M : \sigma]. \end{aligned}$$

In the last line in the equation above we use the Frobenius reciprocity principle, which is described for compact Lie groups in ([Kna86, Theorem 1.14]). By ([Kna86, p.208]), one has

$$[\pi_{\sigma,\lambda}|_K : \tau] = \sum_{\omega \in (M \cap K)^\wedge} n_\omega [\tau|_{M \cup K} : \omega],$$

where n_ω are positive integers. But, in our case $M \subset K$ and therefore $M \cap K = M$. Hence,

$$[\pi_{\sigma,\lambda}|_K : \tau] = [\tau|_M : \sigma].$$

By [GW98, Theorem 8.1.3], K is multiplicity free in G , i.e. $[\pi_{\sigma,\lambda} : \tau] \leq 1$. The assertion follows. \square

We pass now to $X = \Gamma \backslash \tilde{X}$, and we consider the locally homogeneous vector bundle

$$E_\tau := \Gamma \backslash \tilde{E}_\tau \rightarrow X.$$

Let E_χ be the flat vector bundle over X . We want to use the trace formula for the heat operator $e^{-t\Delta_{\tau,\chi}^\#}$, where $\Delta_{\tau,\chi}^\#$ is the twisted Bochner-Laplace operator acting on $C^\infty(X, E_\tau \otimes E_\chi)$.

The key point is that when we consider the lift of the twisted Bochner-Laplace operator to the universal covering, this operator acts as the identity operator on the space of the smooth sections of the flat vector bundle E_χ . We recall here that by formula (4.6),

$$\tilde{\Delta}_{\tau,\chi}^\# = \tilde{\Delta}_\tau \otimes \text{Id}_{V_\chi}.$$

For $\phi \in C(\tilde{X}, \tilde{E}_\tau) \otimes V_\chi$, the unique solution of the heat equation

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \tilde{\Delta}_{\tau,\chi}^\# \right) u^\phi(t; x) &= 0, \\ \lim_{t \rightarrow 0^+} u^\phi(t; x) &= \phi(x) \end{aligned}$$

is

$$u^\phi(t; x) := (e^{-t\tilde{\Delta}_\tau} \otimes \text{Id})\phi(x).$$

This is not difficult to see, since in the Appendix B, Remark B.9, the same statement holds for the heat equation associated with an elliptic bounded and formally self-adjoint operator.

We realize the space of smooth sections of $E_\tau \otimes E_\chi$ as the space of Γ -invariant elements of $C^\infty(\tilde{X}, \tilde{E}_\tau) \otimes V_\chi$, i.e.

$$C^\infty(X, E_\tau \otimes E_\chi) = (C^\infty(\tilde{X}, \tilde{E}_\tau) \otimes V_\chi)^\Gamma.$$

By [Mül11, Lemma 2.4, Proposition 2.5], we conclude that the heat operator $e^{-t\Delta_{\tau,x}^\sharp}$ is an integral trace class operator, whose kernel function is a smooth section of $(E_\tau \otimes E_\chi) \otimes (E_\tau \otimes E_\chi)^*$, i.e.

$$H_t^{\tau,\chi} \in C^\infty(X, (E_\tau \otimes E_\chi) \otimes (E_\tau \otimes E_\chi)^*).$$

We consider now the pullbacks \tilde{x}, \tilde{y} of $x, y \in X$ to \tilde{X} , respectively. Let F be a fundamental domain for Γ . Then, for $f \in (C^\infty(\tilde{X}, \tilde{E}_\tau) \otimes V_\chi)^\Gamma$,

$$\begin{aligned} e^{-t\Delta_{\tau,x}^\sharp} f(x) &= \int_X H_t^{\tau,\chi}(x, y) f(y) dy = \int_{\tilde{X}} (H_t^\tau(\tilde{x}, \tilde{y}) \otimes \text{Id}) f(\tilde{y}) d\tilde{y} \\ &= \sum_{\gamma \in \Gamma} \int_F (H_t^\tau(\tilde{x}, \gamma\tilde{y}) \otimes \chi(\gamma) \text{Id}) f(\tilde{y}) d\tilde{y}. \end{aligned}$$

Here, $H_t^\tau \in (\mathcal{C}^q(G) \otimes \text{End}(V_\tau))^{K \times K}$ as in Theorem 4.2. It corresponds to the kernel of the integral operator $e^{-t\tilde{\Delta}}$ as in (4.14).

Therefore, the kernel function $H_t^{\tau,\chi}(x, y) \in C^\infty(X, (E_\tau \otimes E_\chi) \otimes (E_\tau \otimes E_\chi)^*)$ is given by

$$H_t^{\tau,\chi}(x, y) = \sum_{\gamma \in \Gamma} H_t^\tau(\tilde{x}, \gamma\tilde{y}) \otimes \chi(\gamma) \text{Id}.$$

Hence, we have the following proposition.

Proposition 4.4. *Let E_χ be a flat vector bundle over $X = \Gamma \backslash \tilde{X}$, associated with a finite dimensional complex representation $\chi: \Gamma \rightarrow \text{GL}(V_\chi)$ of Γ . Let $\Delta_{\tau,x}^\sharp$ be the twisted Bochner-Laplace operator acting on $C^\infty(X, E_\tau \otimes E_\chi)$. Then,*

$$\text{Tr}(e^{-t\Delta_{\tau,x}^\sharp}) = \sum_{\gamma \in \Gamma} \text{tr} \chi(\gamma) \int_{\Gamma \backslash G} \text{tr} H_t^\tau(g^{-1}\gamma g) dg, \quad (4.20)$$

where $H_t^\tau \in (\mathcal{C}^q(G) \otimes \text{End}(V_\tau))^{K \times K}$.

We proceed as usual to obtain a better version of the trace formula, analyzing the above identity in orbital integrals. We group together into the conjugacy classes $[\gamma]$ of Γ , and we write separately the conjugacy class of the identity element e . Then, as in Section 3.1 (equation (3.6)) we have

Corollary 4.5.

$$\begin{aligned} \text{Tr}(e^{-t\Delta_{\tau,x}^\sharp}) &= \dim(V_\chi) \text{Vol}(X) \text{tr} H_t^\tau(e) \\ &+ \sum_{[\gamma] \neq e} \text{tr} \chi(\gamma) \text{Vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} \text{tr} H_t^\tau(g^{-1}\gamma g) dg, \end{aligned} \quad (4.21)$$

where Γ_γ and G_γ are the centralizers of γ in Γ and G , respectively.

4.3 The trace formula

We want to construct a $\mathbb{Z}/2$ -graded vector bundle on X . This is because we are interested in analysis on X and specifically in the zeta functions associated with a geodesic flow on the bundle $E(\sigma, \chi) := G \times_{\sigma \otimes \chi} (V_\sigma \otimes V_\gamma) \rightarrow S(X)$ as it is explained in Section 2.1.

We recall from Section 1.1 that for $\tau \in \widehat{K}$ we consider its highest weight ν_τ , given by

$$\nu_\tau = (\nu_1, \dots, \nu_n),$$

where $\nu_1 \geq \dots \geq \nu_n$ and $\nu_i, i = 1, \dots, n$ are all integers or all half integers.

Let now $R(M)^+$ and $R(M)^-$ be the subspaces of the elements of $R(M)$ that are invariant, respectively not invariant, under the action of the Weyl group W_A . More precisely, since the order of the Weyl group W_A is two, there is an eigenspace decomposition of $R(M)$ into $R(M)^+$ and $R(M)^-$. The subspace $R(M)^+$ corresponds to the (+1)-eigenspace, and the subspace $R(M)^-$ corresponds to (-1)-eigenspace with respect to the action of W_A on $R(M)$.

Proposition 4.6. 1. *The map i^* is a bijection between $R(K)$ and $R(M)^+$*

2. *If $\sigma \in \widehat{M}$, then there exists a unique element $\tau(\sigma) \in \widehat{K}$, with highest weight $\nu_\tau = ((\nu_1 - \frac{1}{2})e_1, \dots, (\nu_n - \frac{1}{2})e_n)$ and $\nu_n(\sigma) > 0$, such that*

$$\sigma - w\sigma = (s^+ - s^-)i^*(\tau(\sigma)), \quad (4.22)$$

where s^+, s^- are the half spin-representations of M . More precisely, if s is the spin-representation of K , then $\tau(\sigma) \otimes s$ splits into

$$\tau(\sigma) \otimes s = \tau^+(\sigma) \oplus \tau^-(\sigma) \quad (4.23)$$

such that

$$\sigma + w\sigma = i^*(\tau^+(\sigma) - \tau^-(\sigma)), \quad (4.24)$$

with

$$\tau^\pm(\sigma) = \sum_{\substack{\mu \in \{0,1\}^n \\ c(\mu) = \pm 1}} (-1)^{c(\mu)} \tau_{\nu_\mu}(\sigma), \quad (4.25)$$

where $c(\mu) := \#\{1 \in \mu\}$, $\tau_{\nu_\mu}(\sigma)$ is the representation of K with highest weight $\nu_\mu(\sigma) = \nu_\sigma - \mu$, and ν_σ is given by (1.10).

Proof. See [BO95, Proposition 1.1]. □

Let $\tau_\sigma \in R(K)$ with $\tau_\sigma := \tau^+(\sigma) - \tau^-(\sigma)$. By Proposition 4.6, there exist unique integers $m_\tau(\sigma) \in \{-1, 0, 1\}$, which are equal to zero except for finitely many $\tau \in \widehat{K}$, such that for

- **case (a)**

$$\sigma = \sum_{\tau \in \widehat{K}} m_\tau(\sigma) i^*(\tau); \quad (4.26)$$

- **case (b)**

$$\sigma + w\sigma = \sum_{\tau \in \widehat{K}} m_\tau(\sigma) i^*(\tau). \quad (4.27)$$

Then, the locally homogeneous vector bundle $E(\sigma)$ associated with τ is of the form

$$E(\sigma) = \bigoplus_{\substack{\tau \in \widehat{K} \\ m_\tau(\sigma) \neq 0}} E_\tau, \quad (4.28)$$

where E_τ is the locally homogeneous vector bundle associated with $\tau \in \widehat{K}$. Therefore, the vector bundle $E(\sigma)$ has a grading $E(\sigma) = E(\sigma)^+ \oplus E(\sigma)^-$. This grading is defined exactly by the positive or negative sign of $m_\tau(\sigma)$. Let $\widetilde{E}(\sigma)$ be the pullback of $E(\sigma)$ to \widetilde{X} . Then,

$$\widetilde{E}(\sigma) = \bigoplus_{\substack{\tau \in \widehat{K} \\ m_\tau(\sigma) \neq 0}} \widetilde{E}_\tau.$$

We consider now the lift $\widetilde{\Delta}_\tau$ of the Bochner-Laplace operator Δ_τ associated to $\tau \in \widehat{K}$ to \widetilde{X} , acting on smooth sections of \widetilde{E}_τ . Recall equation (4.11) from Section 4.2,

$$\widetilde{\Delta}_\tau = -R(\Omega) + \lambda_\tau \text{Id}.$$

We put

$$\widetilde{A}_\tau := \widetilde{\Delta}_\tau - \lambda_\tau \text{Id}. \quad (4.29)$$

Hence, by equation (4.11) the operator \widetilde{A}_τ acts like $-R(\Omega)$ on the space of smooth sections of \widetilde{E}_τ . It is an elliptic formally self-adjoint operator of second order. By [Che73] it is an essentially self-adjoint operator. Its self-adjoint closure will be also denoted by \widetilde{A}_τ .

Let $\chi: \Gamma \rightarrow \text{GL}(V_\chi)$ be a finite dimensional complex representation of Γ . Let E_χ be the associated flat vector bundle over X and let \widetilde{E}_χ be the pullback of E_χ to \widetilde{X} .

We use the flat Laplacian $\widetilde{\Delta}_{E_0, \chi}^\sharp$ on the universal covering \widetilde{X} , which is defined in Section 4.1. We get then the operator $\widetilde{A}_{\tau, \chi}^\sharp$ acting on the space $C^\infty(\widetilde{X}, \widetilde{E}_\tau \otimes \widetilde{E}_\chi)$.

Since $\tilde{A}_{\tau,\chi}^\sharp$ is induced by the operator $\tilde{\Delta}_{\tau,\chi}^\sharp$ we get by (4.6) that $\tilde{A}_{\tau,\chi}^\sharp$ can be locally described as

$$\tilde{A}_{\tau,\chi}^\sharp = \tilde{A}_\tau \otimes \text{Id}_{V_\chi}. \quad (4.30)$$

We pass to $X = \Gamma \backslash \tilde{X}$. We put

$$c(\sigma) := -|\rho|^2 - |\rho_m|^2 + |\nu_\sigma + \rho_m|^2, \quad (4.31)$$

where ν_σ is the highest weight of $\sigma \in \widehat{M}$ as in (1.9), and ρ, ρ_m are defined by (1.8) and (1.9), respectively. We define the operator $A_\chi^\sharp(\sigma)$ acting on $C^\infty(X, E(\sigma) \otimes E_\chi)$ by

$$A_\chi^\sharp(\sigma) := \bigoplus_{m_\tau(\sigma) \neq 0} A_{\tau,\chi}^\sharp + c(\sigma). \quad (4.32)$$

Obviously, $A_\chi^\sharp(\sigma)$ preserves the grading.

The operator $A_\chi^\sharp(\sigma)$ is an elliptic operator of order two. However, the situation is now different, because it is not a self-adjoint operator anymore. This property is carried by the operator $\tilde{A}_{\tau,\chi}^\sharp$.

We deal first with the corresponding heat semi-group generated by the operator $e^{-tA_{\tau,\chi}^\sharp}$. Since $A_{\tau,\chi}^\sharp$ is induced by $\Delta_{\tau,\chi}^\sharp$, we have, as in Section 4.2, that

$$e^{-tA_{\tau,\chi}^\sharp} f(x) = \int_X Q_t^{\tau,\chi}(x, y) f(y) dy, \quad (4.33)$$

with

$$Q_t^{\tau,\chi}(x, y) = \sum_{\gamma \in \Gamma} Q_t^\tau(\tilde{x}, \gamma \tilde{y}) \otimes \chi(\gamma) \text{Id},$$

where \tilde{x}, \tilde{y} denote the pullbacks of $x, y \in X$ to \tilde{X} , respectively, and $Q_t^\tau \in (\mathcal{C}^q(G) \otimes \text{End}(V_\tau))^{K \times K}$ is the kernel associated to the operator $e^{-t\tilde{A}_\tau}$. By Proposition 4.4, we get

$$\text{Tr}(e^{-tA_{\tau,\chi}^\sharp}) = \sum_{\gamma \in \Gamma} \text{tr} \chi(\gamma) \int_{\Gamma \backslash G} \text{tr} Q_t^\tau(g^{-1} \gamma g) dg. \quad (4.34)$$

We put

$$q_t^\tau(g) = \text{tr} Q_t^\tau(g).$$

$$q_t^\sigma = \sum_{\tau \in \widehat{K}} m_\tau(\sigma) q_t^\tau, \quad (4.35)$$

$$K(t; \sigma) = \sum_{\tau \in \widehat{K}} m_\tau(\sigma) \text{Tr}(e^{-tA_{\tau,\chi}^\sharp}). \quad (4.36)$$

We use now the trace formula (3.20) in section 3.2. We have

$$K(t; \sigma) = \dim(V_\chi) \text{Vol}(X) q_t^\sigma(e) + \frac{1}{2\pi} \sum_{[\gamma] \neq e} \frac{l(\gamma) \text{tr}(\chi(\gamma))}{n_\Gamma(\gamma) D(\gamma)} \sum_{\sigma \in \widehat{M}} \overline{\text{tr} \sigma(m_\gamma)} \int_{\mathbb{R}} \Theta_{\sigma, \lambda}(q_t^\sigma) e^{-il(\gamma)\lambda} d\lambda. \quad (4.37)$$

We continue analyzing the trace formula above in terms of characters. For the identity contribution we have

$$(q_t^\sigma)(e) = \sum_{\sigma \in \widehat{M}} \int_{\mathbb{R}} \Theta_{\sigma, \lambda}(q_t^\sigma) P_\sigma(i\lambda) d\lambda, \quad (4.38)$$

where $P_\sigma(i\lambda)$ denotes the Plancherel polynomial, defined in Section 3.2. By equation (4.35) we get

$$\Theta_{\sigma, \lambda}(q_t^\sigma) = \sum_{\tau \in \widehat{K}} m_\tau(\sigma) \Theta_{\sigma, \lambda}(q_t^\tau). \quad (4.39)$$

We recall Proposition 4.3 in Section 4.2. Then,

$$\Theta_{\sigma, \lambda}(q_t^\tau) = e^{-t(-\pi_{\sigma, \lambda}(\Omega))} [\tau|_M : \sigma]. \quad (4.40)$$

The term λ_τ does not occur here, since our operator $A_{\tau, \chi}^\sharp$ is induced by the operator $A_\tau = \Delta_\tau - \lambda_\tau \text{Id}$. We recall also

$$\pi_{\sigma, \lambda}(\Omega) = -\lambda^2 + c(\sigma). \quad (4.41)$$

This is proved in [Art, p.48].

Combining equations (4.39), (4.40) and (4.41) we get

$$\Theta_{\sigma, \lambda}(q_t^\sigma) = \sum_{\tau \in \widehat{K}} m_\tau(\sigma) e^{-t(\lambda^2 - c(\sigma))} [\tau|_M : \sigma]. \quad (4.42)$$

Equivalently, for $\sigma, \sigma' \in \widehat{M}$

$$\Theta_{\sigma', \lambda}(q_t^\sigma) = e^{tc(\sigma)} e^{-t\lambda^2} \left[\sum_{\tau \in \widehat{K}} m_\tau(\sigma) i^*(\tau) : \sigma' \right].$$

Hence, by (4.26) and (4.27), we have

$$\begin{aligned} \Theta_{\sigma', \lambda}(q_t^\sigma) &= e^{tc(\sigma)} e^{-t\lambda^2}, \quad \text{if } \sigma' \in \{\sigma, w\sigma\}, \\ \Theta_{\sigma', \lambda}(q_t^\sigma) &= 0, \quad \text{if } \sigma' \notin \{\sigma, w\sigma\}. \end{aligned} \quad (4.43)$$

We put everything together and insert (4.38) and (4.43) in (4.37). Then, we have

- case(a)

$$K(t; \sigma) = e^{tc(\sigma_k)} \left(\dim(V_\chi) \text{Vol}(X) \int_{\mathbb{R}} e^{-t\lambda^2} P_\sigma(i\lambda) d\lambda + \sum_{[\gamma] \neq [e]} \frac{l(\gamma)}{n_\Gamma(\gamma)} L_{sym}(\gamma; \sigma) \frac{e^{-l(\gamma)^2/4t}}{(4\pi t)^{1/2}} \right);$$

- case(b)

$$K(t; \sigma) = e^{tc(\sigma_k)} \left(2 \dim(V_\chi) \text{Vol}(X) \int_{\mathbb{R}} e^{-t\lambda^2} P_\sigma(i\lambda) d\lambda + \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} L_{sym}(\gamma; \sigma + w\sigma) \frac{e^{-l(\gamma)^2/4t}}{(4\pi t)^{1/2}} \right),$$

where

$$L_{sym}(\gamma; \sigma) = \frac{\text{tr}(\chi(\gamma) \otimes \sigma(m_\gamma)) e^{-|\rho|l(\gamma)}}{\det(\text{Id} - \text{Ad}(m_\gamma a_\gamma \bar{\alpha}))}. \quad (4.44)$$

By the definition of the operator $A_\chi^\sharp(\sigma)$ in (4.32), we get the following theorem.

Theorem 4.7 (trace formula for the operator $e^{-tA_\chi^\sharp(\sigma)}$). *For every $\sigma \in \widehat{M}$,*

- case(a)

$$\text{Tr}(e^{-tA_\chi^\sharp(\sigma)}) = \dim(V_\chi) \text{Vol}(X) \int_{\mathbb{R}} e^{-t\lambda^2} P_\sigma(i\lambda) d\lambda + \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} L_{sym}(\gamma; \sigma) \frac{e^{-l(\gamma)^2/4t}}{(4\pi t)^{1/2}}; \quad (4.45)$$

- case(b)

$$\text{Tr}(e^{-tA_\chi^\sharp(\sigma)}) = 2 \dim(V_\chi) \text{Vol}(X) \int_{\mathbb{R}} e^{-t\lambda^2} P_\sigma(i\lambda) d\lambda + \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} L_{sym}(\gamma; \sigma + w\sigma) \frac{e^{-l(\gamma)^2/4t}}{(4\pi t)^{1/2}}, \quad (4.46)$$

where $L_{sym}(\gamma; \sigma)$ is as in (4.44).

CHAPTER 5

The twisted Dirac operator

5.1 Dirac operators

Let $\sigma \in \widehat{M}$ be an irreducible representation of M with highest weight ν_σ as in (1.9). We recall that ν_n denotes the last coordinate of ν_σ . Throughout this section we will consider $\sigma \in R(M)^-$, i.e., we examine the case (b), where σ is not invariant under the action of W_A . We recall also from Section 1.1,

$$\begin{aligned} K &= \text{Spin}(d), \\ M &= \text{Spin}(d-1), \end{aligned}$$

where d is an odd integer.

Let s be the spin representation of K . Since $d-1$ is an even integer, s splits into two irreducible half-spin representations (s^+, S^+) , (s^-, S^-) of M , as in Section 1.1. Let $\text{Cl}(\mathfrak{p})$ be the Clifford algebra of \mathfrak{p} with respect to the inner product $\langle \cdot, \cdot \rangle_0$, as in (1.4), restricted to \mathfrak{p} . Let

$$\cdot : \mathfrak{p} \otimes S \rightarrow S$$

be the Clifford multiplication on $\mathfrak{p} \otimes S$. Let H be in \mathfrak{a}^+ , where \mathfrak{a}^+ is the Lie algebra of A^+ and A^+ is defined as in (1.7). Since M centralizes \mathfrak{a} , the Clifford multiplication by H preserves the decomposition $S = S^+ \oplus S^-$. The Clifford multiplication by H^2 acts as $-\text{Id}$. Then, H acts on S^\pm with eigenvalues $\pm i$. Hence, we can consider the Clifford multiplication by H as multiplication by $\pm i \text{sign}(\nu_n)$. We consider the connection ∇ in $\text{Cl}(\mathfrak{p})$, induced by the canonical connection in the tangent frame bundle of X . Let L be any bundle of left modules over $\text{Cl}(\mathfrak{p})$ over \widetilde{X} , i.e. a spinor bundle over \widetilde{X} . We lift the connection ∇ in L and we obtain

a connection also denoted by ∇ . The Dirac operator $D: C^\infty(X, L) \rightarrow C^\infty(X, L)$ is defined as

$$D: C^\infty(X, L) \xrightarrow{\nabla} C^\infty(X, T^*X \otimes L) \xrightarrow{g} C^\infty(X, TX \otimes L) \rightarrow C^\infty(X, L),$$

where we identify $TX \cong T^*X$ using the riemannian metric g , and \cdot denotes the Clifford multiplication as above.

Locally, it can be described as

$$Df \equiv \sum_{i=1}^d e_i \cdot \nabla_{e_i} f,$$

where (e_1, \dots, e_d) is a local orthonormal frame for $T_x X, x \in X$.

The bundle L is a Dirac bundle over \tilde{X} . This means that

- the Clifford multiplication by unit vectors in $\text{Cl}(\mathfrak{p})$ is orthogonal i.e.

$$\langle ef_1, ef_2 \rangle = \langle f_1, f_2 \rangle,$$

for all unit vectors $e \in T_x \tilde{X}, x \in \tilde{X}$ and all $f_1, f_2 \in L_x$, where L_x denotes the fiber of L over $x \in \tilde{X}$.

- the connection ∇ satisfies the product rule

$$\nabla(\phi f) = (\nabla\phi) \cdot f + \phi \cdot (\nabla f),$$

for all $\phi \in C^\infty(X, \text{Cl}(\mathfrak{p}))$ and all $f \in C^\infty(X, L)$.

The operator D is an elliptic ([LM89, Lemma 5.1]), formally self-adjoint ([LM89, Proposition 5.3]) operator of first order.

We want to define *twisted* Dirac operators acting on smooth sections of vector bundles associated with the representations σ of M and arbitrary representations χ of Γ .

Proposition 5.1. *Let $\sigma \in \widehat{M}$. Then, there exists a unique element $\tau(\sigma) \in \widehat{K}$ and a splitting*

$$s \otimes \tau(\sigma) = \tau^+(\sigma) \oplus \tau^-(\sigma)$$

where $\tau^+(\sigma), \tau^-(\sigma) \in R(K)$ such that

$$\sigma + w\sigma = i^*(\tau^+(\sigma) - \tau^-(\sigma)) \tag{5.1}$$

Proof. This is proved in [BO95, Proposition 1.1, (3)]. □

We define the representation $\tau_s(\sigma)$ of K by

$$\tau_s(\sigma) := s \otimes \tau(\sigma), \quad (5.2)$$

with representation space $V_{\tau_s(\sigma)} = S \otimes V_{\tau(\sigma)}$, where $V_{\tau(\sigma)}$ is the representation space of $\tau(\sigma)$.

We consider the homogeneous vector bundle $\tilde{E}_{\tau(\sigma)}$ over \tilde{X} given by

$$\tilde{E}_{\tau(\sigma)} = G \times_{\tau(\sigma)} V_{\tau(\sigma)} \rightarrow \tilde{X}.$$

The vector bundle $\tilde{E}_{\tau_s(\sigma)} := \tilde{E}_{\tau(\sigma)} \otimes S$ over \tilde{X} carries a connection $\nabla^{\tau_s(\sigma)}$, defined by the formula

$$\nabla^{\tau_s(\sigma)} = \nabla^{\tau(\sigma)} \otimes 1 + 1 \otimes \nabla.$$

where $\nabla^{\tau(\sigma)}$ denotes the canonical connection in $\tilde{E}_{\tau(\sigma)}$.

We extend the Clifford multiplication by requiring that it acts on $V_{\tau_s(\sigma)} = S \otimes V_{\tau(\sigma)}$ as follows.

$$e \cdot (\phi \otimes \psi) = (e \cdot \phi) \otimes \psi, \quad e \in \text{Cl}(\mathfrak{p}), \phi \in S, \psi \in V_{\tau(\sigma)}.$$

We define the Dirac operator $\tilde{D}(\sigma)$ acting on $C^\infty(\tilde{X}, V_{\tau_s(\sigma)})$ by

$$\tilde{D}(\sigma)f = \sum_{i=1}^d e_i \cdot \nabla_{e_i}^{\tau_s(\sigma)} f,$$

where (e_1, \dots, e_d) is local orthonormal frame for $T_x \tilde{X}$ and $f \in C^\infty(\tilde{X}, V_{\tau_s(\sigma)})$. The space of smooth sections $C^\infty(\tilde{X}, V_{\tau_s(\sigma)})$ can be identified with $C^\infty(G; \tau_s(\sigma))$ as in (4.10)

Let now $\chi : \Gamma \rightarrow \text{GL}(V_\chi)$ be an arbitrary finite dimensional representation of Γ . Let E_χ be the associated flat vector bundle over X . Let $E_{\tau_s(\sigma)} := \Gamma \backslash \tilde{E}_{\tau_s(\sigma)}$ be the locally homogeneous vector bundle over X . We consider the product vector bundle $E_{\tau_s(\sigma)} \otimes E_\chi$ over X and we equip this bundle with the product connection $\nabla^{E_{\tau_s(\sigma)} \otimes E_\chi}$ defined by

$$\nabla^{E_{\tau_s(\sigma)} \otimes E_\chi} = \nabla^{E_{\tau_s(\sigma)}} \otimes 1 + 1 \otimes \nabla^{E_\chi}.$$

We consider the Clifford multiplication on $(V_{\tau_s(\sigma)} \otimes V_\chi)$ by requiring that it acts only on $V_{\tau_s(\sigma)}$, i.e.

$$e \cdot (w \otimes v) = (e \cdot w) \otimes v, \quad e \in \text{Cl}(\mathfrak{p}), w \in V_{\tau_s(\sigma)}, v \in V_\chi.$$

For our proposal we introduce the twisted Dirac operator $D_\chi^\sharp(\sigma)$ associated with $\nabla^{E_{\tau_s(\sigma)} \otimes E_\chi}$. We want to describe it locally. We consider then an open subset of X such that $E_\chi|_U$ is trivial. Let also $(v_j), j = 1, \dots, m$, be a basis of flat sections of $E_\chi|_U$, where $m = \text{rank}(E_\chi)$, and $\phi_j \in C^\infty(U, E_{\tau_s(\sigma)}|_U)$. Then,

$$E_{\tau_s(\sigma)} \otimes E_\chi|_U \cong \bigoplus_{j=1}^m E_{\tau_s(\sigma)}|_U,$$

and for each $\phi \in C^\infty(U, E_{\tau_s(\sigma)} \otimes E_\chi|_U)$,

$$\phi = \sum_{j=1}^m \phi_j \otimes v_j.$$

The product connection is given by

$$\nabla^{E_{\tau_s(\sigma)} \otimes E_\chi}(\phi) = \sum_{j=1}^m (\nabla^{E_{\tau_s(\sigma)}})(\phi_j) \otimes v_j.$$

Then the Dirac operator is described as follows

$$\begin{aligned} D_\chi^\sharp(\sigma)\phi &= \sum_{i=1}^d e_i \cdot \nabla_{e_i}^{E_{\tau_s(\sigma)} \otimes E_\chi}(\phi) \\ &= \sum_{i=1}^d e_i \cdot \left(\sum_{j=1}^m (\nabla_{e_i}^{E_{\tau_s(\sigma)}})(\phi_j) \otimes v_j \right) \\ &= \sum_{i=1}^d \sum_{j=1}^m e_i \cdot ((\nabla_{e_i}^{E_{\tau_s(\sigma)}})(\phi_j) \otimes v_j). \end{aligned} \quad (5.3)$$

We consider the pullbacks $\tilde{E}_{\tau_s(\sigma)}, \tilde{E}_\chi$ to \tilde{X} of $E_{\tau_s(\sigma)}, E_\chi$, respectively, then, $\tilde{E}_\chi \cong \tilde{X} \times V_\chi$. We have

$$C(\tilde{X}, \tilde{E}_{\tau_s(\sigma)} \otimes \tilde{E}_\chi) \cong C(\tilde{X}, \tilde{E}_{\tau_s(\sigma)}) \otimes V_\chi.$$

With respect to this isomorphism, it follows from (5.3) that the lift $\tilde{D}_\chi^\sharp(\sigma)$ of the twisted Dirac operator $D_\chi^\sharp(\sigma)$ to \tilde{X} is of the form

$$\tilde{D}_\chi^\sharp(\sigma) = \tilde{D}(\sigma) \otimes \text{Id}_{V_\chi}. \quad (5.4)$$

5.2 The trace formula

The square of the twisted Dirac operator $(D_\chi^\sharp(\sigma))^2$ acting on smooth sections of $E_{\tau_s(\sigma)} \otimes E_\chi$ is not a self-adjoint operator in general. Nevertheless, its principal symbol is given by

$$\sigma_{(D_\chi^\sharp(\sigma))^2}(x, \xi) = \|\xi\|^2 \otimes \text{Id}_{(V_{\tau_s(\sigma)} \otimes V_\chi)_x}, \quad x \in X, \quad \xi \in T_x^*X, \xi \neq 0.$$

Therefore, it has nice spectral properties. We recall from Appendix A, Lemma A.10 that its spectrum is discrete and contained in a translate of a positive cone $C \subset \mathbb{C}$.

Let $\tilde{D}_\chi^\sharp(\sigma)$ be the lift of $D_\chi^\sharp(\sigma)$ to the universal covering \tilde{X} . We recall equation (5.4),

$$\tilde{D}_\chi^\sharp(\sigma) = \tilde{D}(\sigma) \otimes \text{Id}_{V_\chi}.$$

Hence,

$$(\tilde{D}_\chi^\sharp(\sigma))^2 = (\tilde{D}(\sigma))^2 \otimes \text{Id}_{V_\chi} \quad (5.5)$$

We recall the definition of the operator $A_\chi^\sharp(\sigma)$ acting on $C^\infty(X, E(\sigma) \otimes E_\chi)$ from Section 4.3.

$$A_\chi^\sharp(\sigma) := \bigoplus_{m_\tau(\sigma) \neq 0} A_{\tau, \chi}^\sharp + c(\sigma),$$

where $c(\sigma)$ is as in (4.31). We consider the lift $\tilde{A}_\chi^\sharp(\sigma)$ of $A_\chi^\sharp(\sigma)$ to the universal covering \tilde{X} . Then,

$$\begin{aligned} \tilde{A}_\chi^\sharp(\sigma) &= \bigoplus_{m_\tau(\sigma) \neq 0} \tilde{A}_{\tau, \chi}^\sharp + c(\sigma) \\ &= \bigoplus_{m_\tau(\sigma) \neq 0} (\tilde{A}_\tau \otimes \text{Id}_{V_\chi}) + c(\sigma) \\ &= \bigoplus_{m_\tau(\sigma) \neq 0} (\tilde{A}_\tau + c(\sigma)) \otimes \text{Id}_{V_\chi}. \end{aligned} \quad (5.6)$$

The Parthasarathy formula from [BO95, eq.(1.11)] states

$$(\tilde{D}(\sigma))^2 = \bigoplus_{m_\tau(\sigma) \neq 0} (\tilde{A}_\tau + c(\sigma)) \quad (5.7)$$

If we combine (5.5), (5.6) and (5.7) the Parthasarathy formula generalizes as

$$(D_\chi^\sharp(\sigma))^2 = A_\chi^\sharp(\sigma). \quad (5.8)$$

We define the operator $D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2}$ by equation (A.5):

$$D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2} = \frac{i}{2\pi} \int_{\Gamma_{\theta, r_0}} \lambda^{1/2} e^{-t\lambda} ((D_\chi^\sharp(\sigma))^2 - \lambda \text{Id})^{-1} d\lambda,$$

from Appendix A. As in [Mül11, Lemma 2.4], we conclude that $D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2}$ is an integral operator. Let $K_t^{\tau_s(\sigma), \chi}$ be its kernel function. Let F be a fundamental domain of Γ . We consider the space $L^2(\tilde{X}, \tilde{E}_{\tau_s(s)} \otimes \tilde{E}_\chi)^\Gamma$ of sections f of $\tilde{E}_{\tau_s(s)} \otimes \tilde{E}_\chi$ such that $f(\gamma\tilde{x}) = \chi(\gamma)f(\tilde{x})$, $\forall \gamma \in \Gamma, \tilde{x} \in \tilde{X}$.

For $f \in L^2(X, E_{\tau_s(s)} \otimes E_\chi) \cong L^2(\tilde{X}, \tilde{E}_{\tau_s(s)} \otimes \tilde{E}_\chi)^\Gamma$, we have

$$\begin{aligned} D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2} f(x) &= \int_X K_t^{\tau_s(\sigma), \chi}(x, y) f(y) dy \\ &= \int_{\tilde{X}} (K_t^{\tau_s(\sigma)}(\tilde{x}, \tilde{y}) \otimes \text{Id}_{V_\chi}) f(\tilde{y}) d\tilde{y} \\ &= \sum_{\gamma \in \Gamma} \int_F (K_t^{\tau_s(\sigma)}(\tilde{x}, \gamma\tilde{y}) \otimes \chi(\gamma) \text{Id}_{V_\chi}) f(\tilde{y}) d\tilde{y}, \end{aligned} \quad (5.9)$$

where $x, y \in X$ and $\tilde{x}, \tilde{y} \in \tilde{X}$ are lifts of x, y to \tilde{X} , respectively. The kernel function $K_t^{\tau_s(\sigma)}$ is the kernel associated with the operator $\tilde{D}(\sigma)e^{-t(\tilde{D}(\sigma))^2}$. It belongs to the Harish-Chandra L^q -Schwartz space $(\mathcal{C}^q(G) \otimes \text{End}(V_{\tau_s(\sigma)}))^{K \times K}$. Hence, we can interchange summation and integration in the right hand side of (5.9) and get

$$K_t^{\tau_s(\sigma), \chi}(x, x') = \sum_{\gamma \in \Gamma} K_t^{\tau_s(\sigma)}(g^{-1}\gamma g') \otimes \chi(\gamma),$$

where $x = \Gamma g, x' = \Gamma g', g, g' \in G$.

By [Mül11, Proposition 2.5], $D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2}$ is a trace class operator, and its trace is given by

$$\text{Tr}(D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2}) = \sum_{\gamma \in \Gamma} \text{tr} \chi(\gamma) \int_{\Gamma \backslash G} \text{tr} K_t^{\tau_s(\sigma)}(g^{-1}\gamma g) dg.$$

We put

$$k_t^{\tau_s(\sigma)}(g) = \text{tr} K_t^{\tau_s(\sigma)}(g). \quad (5.10)$$

We use the trace formula (3.20) from Section 3.2,

$$\begin{aligned} \text{Tr}(D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2}) &= \dim(V_\chi) \text{Vol}(X) (k_t^{\tau_s(\sigma)})(e) \\ &+ \frac{1}{2\pi} \sum_{[\gamma] \neq e} \frac{l(\gamma) \text{tr}(\chi(\gamma))}{n_\Gamma(\gamma) D(\gamma)} \sum_{\sigma \in \tilde{M}} \overline{\text{tr} \sigma(m_\gamma)} \int_{\mathbb{R}} \Theta_{\sigma, \lambda}(k_t^{\tau_s(\sigma)}) e^{-il(\gamma)\lambda} d\lambda. \end{aligned}$$

We continue analyzing the trace formula above in terms of characters. We want to compute the *Fourier transform* $\Theta_{\sigma,\lambda}(k_t^{\tau_s(\sigma)})$ of $k_t^{\tau_s(\sigma)}$. Following [MS89] we let (π, \mathcal{H}_π) be an unitary admissible representation of G in a Hilbert space \mathcal{H}_π . We let \mathcal{H}_π^∞ be the subspace of smooth vectors of \mathcal{H}_π . We set

$$\pi(K_t^{\tau_s(\sigma)}) := \int_G \pi(g) \otimes K_t^{\tau_s(\sigma)}(g) dg. \quad (5.11)$$

This defines a bounded trace class operator on $\mathcal{H}_\pi \otimes V_{\tau_s(\sigma)}$. Then, as in Section 4.2 (equation (4.17)),

$$\mathrm{Tr}(\pi(k_t^{\tau_s(\sigma)})) = \mathrm{Tr}(\tilde{\pi}(K_t^{\tau_s(\sigma)})). \quad (5.12)$$

Let $(X_i)_{i=1}^d$ be an orthonormal basis of \mathfrak{p} . We consider the operator acting on $(\mathcal{H}_\pi^\infty \otimes V_{\tau_s(\sigma)})^K$, defined by

$$\tilde{D}_{\tau_s(\sigma)}(\pi) := \sum_{i=1}^d X_i \cdot (\pi(X_i) \otimes \mathrm{Id}). \quad (5.13)$$

In [Pfa12, p.77], it is proved that $\tilde{D}_{\tau_s(\sigma)}(\pi)$ maps $(\mathcal{H}_\pi^\infty \otimes V_{\tau_s(\sigma)})^K$ to $(\mathcal{H}_\pi^\infty \otimes V_{\tau_s(\sigma)})^K$. By (5.11) we have

$$\tilde{\pi}(K_t^{\tau_s(\sigma)}) = e^{-tc(\sigma)} \tilde{D}_{\tau_s(\sigma)}(\pi) \circ \tilde{\pi}(H_t^{\tau_s(\sigma)}), \quad (5.14)$$

where

$$\tilde{\pi}(H_t^{\tau_s(\sigma)}) = \int_G \pi(g) \otimes H_t^{\tau_s(\sigma)} dg.$$

The kernel function $H_t^{\tau_s(\sigma)}$ corresponds to the integral operator

$$e^{-t(\tilde{D}(\sigma))^2} f(g) = e^{-tc(\sigma)} \int_G H_t^{\tau_s(\sigma)}(g^{-1}g') f(g') dg',$$

where $(\tilde{D}(\sigma))^2$ as in (5.7). We have that $H_t^{\tau_s(\sigma)}$ belongs to the Harish-Chandra L^q -Schwartz space $(\mathcal{C}^q(G) \otimes \mathrm{End}(V_{\tau_s(\sigma)}))^{K \times K}$.

Similar to Section 4.2, we have for the operator $\tilde{\pi}(H_t^{\tau_s(\sigma)})$ that relative to the splitting,

$$\mathcal{H}_\pi \otimes V_{\tau_s(\sigma)} = (\mathcal{H}_\pi \otimes V_{\tau_s(\sigma)})^K \oplus [(\mathcal{H}_\pi \otimes V_{\tau_s(\sigma)})^K]^\perp,$$

it takes the form

$$\tilde{\pi}(H_t^{\tau_s(\sigma)}) = \begin{pmatrix} \pi(H_t^{\tau_s(\sigma)}) & 0 \\ 0 & 0 \end{pmatrix}, \quad (5.15)$$

with $\pi(H_t^{\tau_s(\sigma)})$ acting on $(\mathcal{H}_\pi \otimes V_{\tau_s(\sigma)})^K$. Then, it follows that

$$e^{t\pi(\Omega)} \mathrm{Id} = \pi(H_t^{\tau_s(\sigma)}), \quad (5.16)$$

where Id denotes the identity on the space $(\mathcal{H}_\pi^\infty \otimes V_{\tau_s(\sigma)})^K$ ([BM83, Corollary 2.2]). We have $(\mathcal{H}_\pi \otimes V_{\tau_s(\sigma)})^K = (\mathcal{H}_\pi^\infty \otimes V_{\tau_s(\sigma)})^K$ and

$$\text{Tr}(\pi(k_t^{\tau_s(\sigma)})) = e^{(\pi(\Omega) - c(\sigma))t} \text{Tr}(\tilde{D}_{\tau_s(\sigma)}(\pi)|_{(\mathcal{H}_\pi^\infty \otimes V_{\tau_s(\sigma)})^K}). \quad (5.17)$$

We recall that the representation space of $\tau_s(\sigma)$ is given by

$$V_{\tau_s(\sigma)} = V_{\tau(\sigma)} \otimes S.$$

Let π be the unitary principal series representation $\pi_{\sigma, \lambda}$ defined as in Section 3.2. By [MS89, Proposition 3.6] we have for $(\sigma', V_{\sigma'}) \in \widehat{M}$,

$$\text{Tr}(\tilde{D}_{\tau_s(\sigma)}(\pi_{\sigma', \lambda})) = \lambda(\dim(V_{\sigma'} \otimes V_{\tau(\sigma)} \otimes S^+)^M - \dim(V_{\sigma'} \otimes V_{\tau(\sigma)} \otimes S^-)^M). \quad (5.18)$$

Following [Pfa12, Corollary 7.6] we let $\check{\sigma}'$ be the contragredient representation of σ' . Since $\check{\sigma}' \cong \sigma'$, we observe by equation (5.1) in Proposition 5.1 that for $\nu_n > 0$,

$$[\dim(V_{\sigma'} \otimes V_{\tau(\sigma)} \otimes S^+)^M - \dim(V_{\sigma'} \otimes V_{\tau(\sigma)} \otimes S^-)^M] = [\sigma - w\sigma : \sigma']. \quad (5.19)$$

Since $\sigma' \in \widehat{M}$ we have that $\sigma' \in \{\sigma, w\sigma\}$, otherwise the right hand side of (5.19) vanishes. If we put together (4.43), (5.17), (5.18) and (5.19) we obtain

$$\Theta_{\sigma', \lambda}(k_t^{\tau_s(\sigma)}) = \lambda e^{-t\lambda^2}, \quad \text{if } \sigma' = \sigma \quad (5.20)$$

$$\Theta_{\sigma', \lambda}(k_t^{\tau_s(\sigma)}) = -\lambda e^{-t\lambda^2}, \quad \text{if } \sigma' = w\sigma \quad (5.21)$$

$$\Theta_{\sigma', \lambda}(k_t^{\tau_s(\sigma)}) = 0, \quad \text{if } \sigma' \notin \{\sigma, w\sigma\}. \quad (5.22)$$

For the identity contribution we use the fact that when s is restricted to M it decomposes as $s^+ + s^-$. Furthermore s^+ and s^- are connected by the relation $s^- = ws^+$. The Plancherel polynomial is an even polynomial of λ (cf. section 3.2) and also $P_{s^+}(i\lambda) = P_{ws^+}(-i\lambda) = P_{s^-}(i\lambda)$. Hence,

$$\begin{aligned} k_t^{\tau_s(\sigma)}(e) &= \sum_{\sigma \in \widehat{M}} \int_{\mathbb{R}} \Theta_{\sigma, \lambda}(k_t^{\tau_s(\sigma)}) P_\sigma(i\lambda) d\lambda \\ &= \int_{\mathbb{R}} \lambda e^{-t\lambda^2} P_{s^+}(i\lambda) d\lambda + \int_{\mathbb{R}} -\lambda e^{-t\lambda^2} P_{s^-}(i\lambda) d\lambda = 0. \end{aligned} \quad (5.23)$$

For the hyperbolic contribution we use (5.20)—(5.22).

$$\text{Tr}(D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) = \frac{1}{2\pi} \sum_{[\gamma] \neq e} \frac{l(\gamma) \text{tr}(\chi(\gamma) \otimes (\sigma(m_\gamma) - w\sigma(m_\gamma)))}{D(\gamma)n_\Gamma(\gamma)} \int_{\mathbb{R}} \lambda e^{-t\lambda^2} e^{-il(\gamma)\lambda} d\lambda.$$

Equivalently,

$$\text{Tr}(D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) = \sum_{[\gamma] \neq e} \frac{-2\pi i}{(4\pi t)^{3/2}} \frac{l^2(\gamma) \text{tr}(\chi(\gamma) \otimes (\sigma(m_\gamma) - w\sigma(m_\gamma)))}{n_\Gamma(\gamma) D(\gamma)} e^{-l^2(\gamma)/4t}.$$

All in all, we have proved the following theorem.

Theorem 5.2 (trace formula for the operator $D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2}$). For every $\sigma \in \widehat{M}$,

$$\mathrm{Tr}(D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2}) = \sum_{[\gamma] \neq e} \frac{-2\pi i}{(4\pi t)^{3/2}} \frac{l^2(\gamma) \mathrm{tr}(\chi(\gamma) \otimes (\sigma(m_\gamma) - w\sigma(m_\gamma)))}{n_\Gamma(\gamma)D(\gamma)} e^{-l^2(\gamma)/4t}. \quad (5.24)$$

5.3 The eta function associated with the twisted Dirac operator

In this section we recall at first the definition of the eta function of the twisted Dirac operator $D_\chi^\sharp(\sigma)$. It is important for the proof of the functional equations of the Selberg zeta function to derive a formula that connects the eta invariant $\eta(0, D_\chi^\sharp(\sigma))$ and the trace $\mathrm{Tr}(D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2})$. In the following definition, we use the notion of an Agmon angle and the discreteness of the spectrum of the twisted Dirac operator, which are explained in detail in Appendix A.

Definition 5.3. The angle θ is an Agmon angle for an elliptic operator D , if it is a principal angle for D (cf. Definition A.3., Appendix A) and there exists an $\varepsilon > 0$ such that

$$\mathrm{spec}(D) \cap L_{[\theta-\varepsilon, \theta+\varepsilon]} = \emptyset,$$

where L_I is a solid angle defined by

$$L_I := \{\rho e^{i\theta} : \rho \in (0, \infty), \theta \in I \subset [0, 2\pi]\}.$$

Definition 5.4. Let θ be an Agmon angle for $D_\chi^\sharp(\sigma)$ and let $\mathrm{spec}(D_\chi^\sharp(\sigma)) = \{\lambda_k : k \in \mathbb{N}\}$ be the spectrum of $D_\chi^\sharp(\sigma)$, contained in some discrete subset of \mathbb{C} . Let $m_k = m(\lambda_k)$ be the algebraic multiplicity of the eigenvalue λ_k (cf. Definition A.9., Appendix A). Then, for $\mathrm{Re}(s) > 0$, we define the eta function $\eta_\theta(s, D_\chi^\sharp(\sigma))$ of $D_\chi^\sharp(\sigma)$ by the formula

$$\eta_\theta(s, D_\chi^\sharp(\sigma)) = \sum_{\mathrm{Re}(\lambda_k) > 0} m_k(\lambda_k)_\theta^{-s} - \sum_{\mathrm{Re}(\lambda_k) < 0} m_k(-\lambda_k)_\theta^{-s}. \quad (5.25)$$

It has been shown by [GS95, Theorem 2.7] that $\eta_\theta(s, D_\chi^\sharp(\sigma))$ has a meromorphic continuation to the whole complex plane \mathbb{C} with isolated simple poles and is regular at $s = 0$. Moreover, the number $\eta_\theta(0, D_\chi^\sharp(\sigma))$ is independent of the Agmon angle θ . We call the number $\eta_\theta(0, D_\chi^\sharp(\sigma)) = \eta(0, D_\chi^\sharp(\sigma))$ the *eta invariant* associated with the operator $D_\chi^\sharp(\sigma)$.

We give here a short description of the proof. Definition 5.4 can be read also as

$$\eta_\theta(s, D_\chi^\sharp(\sigma)) = \zeta_\theta(s, \Pi_>, D_\chi^\sharp(\sigma)) - \zeta_\theta(s, \Pi_<, D_\chi^\sharp(\sigma)),$$

where $\Pi_{>}$ (resp. $\Pi_{<}$) is the pseudo-differential projection whose image contains the span of all generalized eigenvectors of $D_{\chi}^{\sharp}(\sigma)$ corresponding to eigenvalues λ with $\operatorname{Re}(\lambda) > 0$ (resp. $\operatorname{Re}(\lambda) < 0$) (For more details see [BK07, Definition 6.16]). The zeta function $\zeta_{\theta}(s, \Pi_{>}, D_{\chi}^{\sharp}(\sigma))$ is define for $\operatorname{Re}(s) > d$,

$$\zeta_{\theta}(s, \square, D_{\chi}^{\sharp}(\sigma)) := \operatorname{Tr}(\square D_{\chi}^{\sharp}(\sigma)^{-s}), \quad (5.26)$$

where $\square = \Pi_{>}, \Pi_{<}$ (cf. [BK07, p.24] and Definition A.6). Then, the meromorphic continuation of the eta function arises from the meromorphic continuation of the kernel of the operator $\square D_{\chi}^{\sharp}(\sigma)^{-s}$ and in case we consider additional $\operatorname{Re}(\lambda) > 0$, from the meromorphic continuation of the kernel of the operator $\square e^{-t(D_{\chi}^{\sharp}(\sigma))^2}$. These operators are defined as follows. We put

$$e^{-t(D_{\chi}^{\sharp}(\sigma))^2} = \frac{i}{2\pi} \int_{\Gamma_{\theta, r_0}} e^{-t\lambda} ((D_{\chi}^{\sharp}(\sigma))^2 - \lambda \operatorname{Id})^{-1} d\lambda,$$

where Γ_{θ, r_0} is the contour defined by $\Gamma_{\theta, r_0} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and $\Gamma_1 = \{-1 + re^{i\theta} : \infty > r \geq r_0\}$, $\Gamma_2 = \{-1 + r_0 e^{ia} : \theta \leq a \leq \theta + 2\pi\}$, $\Gamma_3 = \{-1 + re^{i(\theta+2\pi)} : r_0 \leq r < \infty\}$. On Γ_1 , r runs from ∞ to r_0 , Γ_2 is oriented counterclockwise, and on Γ_3 , r runs from r_0 to ∞ (cf. Appendix A, p.141 and Figure A.2 on page 148).

To define the operator $D_{\chi}^{\sharp}(\sigma)^{-s}$, one has to use the contour Γ_{α, ρ_0} , described as in [Shu87, p.88]. Let α be an Agmon angle for $D_{\chi}^{\sharp}(\sigma)$. We assume that 0 is not an eigenvalue of $D_{\chi}^{\sharp}(\sigma)$. Then, there exists a $\rho_0 > 0$ such that

$$\operatorname{spec}(D) \cap \{z \in \mathbb{C} : |z| \leq 2\rho_0\} = \emptyset.$$

We consider the contour $\Gamma_{\alpha, \rho_0} \subset \mathbb{C}$, defined as $\Gamma_{\alpha, \rho_0} = \Gamma'_1 \cup \Gamma'_2 \cup \Gamma'_3$, where $\Gamma'_1 = \{re^{i\alpha} : \infty > r \geq \rho_0\}$, $\Gamma'_2 = \{\rho_0 e^{i\beta} : \alpha \leq \beta \leq \alpha - 2\pi\}$, $\Gamma'_3 = \{re^{i(\alpha-2\pi)} : \rho_0 \leq r < \infty\}$ (cf. Appendix A, p.141). Then, for $\operatorname{Re}(s) > 0$ we define

$$D_{\chi}^{\sharp}(\sigma)^{-s} = \frac{i}{2\pi} \int_{\Gamma_{\alpha, \rho_0}} \lambda^{-s} (D_{\chi}^{\sharp}(\sigma) - \lambda \operatorname{Id})^{-1} d\lambda.$$

If we integrate by parts the integral above, the operator $(D_{\chi}^{\sharp}(\sigma) - \lambda \operatorname{Id})^{-k}$ will occur. By ([GS95, Theorem 2.7.]), for $k < -d$, there exists an asymptotic expansion of the trace of the operator $\square (D_{\chi}^{\sharp}(\sigma) - \lambda \operatorname{Id})^{-k}$ as $|\lambda| \rightarrow \infty$:

$$\operatorname{Tr}(\square (D_{\chi}^{\sharp}(\sigma) - \lambda \operatorname{Id})^{-k}) \sim \sum_{j=1}^{\infty} c_j \lambda^{d-j-k} + \sum_{l=1}^{\infty} (c'_l \log \lambda + c''_l) \lambda^{-k-l},$$

where the coefficients c_j and c'_l are determined from the symbols of $D_{\chi}^{\sharp}(\sigma)$ and \square , and the coefficients c''_l are in general globally determined.

Let Π_+ be the projection on the span of the root spaces corresponding to eigenvalues λ with $\operatorname{Re}(\lambda^2) > 0$. We consider the Agmon angle θ fixed and we write $\eta_\theta(s, D_\chi^\sharp(\sigma))$ instead of $\eta(s, D_\chi^\sharp(\sigma))$.

We define the functions

$$\begin{aligned}\eta_0(s, D_\chi^\sharp(\sigma)) &:= \sum_{\substack{\operatorname{Re}(\lambda) > 0 \\ \operatorname{Re}(\lambda^2) \leq 0}} \lambda^{-s} - \sum_{\substack{\operatorname{Re}(\lambda) < 0 \\ \operatorname{Re}(\lambda^2) \leq 0}} \lambda^{-s} \\ \eta_1(s, D_\chi^\sharp(\sigma)) &:= \sum_{\substack{\operatorname{Re}(\lambda) > 0 \\ \operatorname{Re}(\lambda^2) > 0}} \lambda^{-s} - \sum_{\substack{\operatorname{Re}(\lambda) < 0 \\ \operatorname{Re}(\lambda^2) > 0}} \lambda^{-s}.\end{aligned}$$

By Definition 5.4, the eta function $\eta(s, D_\chi^\sharp(\sigma))$ satisfies the equation

$$\eta(s, D_\chi^\sharp(\sigma)) = \eta_0(s, D_\chi^\sharp(\sigma)) + \eta_1(s, D_\chi^\sharp(\sigma))$$

Since the spectrum of $(D_\chi^\sharp(\sigma))^2$ is discrete and contained in a translate of a positive cone in \mathbb{C} (cf. Figure 5.1), there are only finitely many eigenvalues of $(D_\chi^\sharp(\sigma))^2$ with $\operatorname{Re}(\lambda^2) \leq 0$.

Lemma 5.5. *The eta function $\eta(s, D_\chi^\sharp(\sigma))$ satisfies the equation*

$$\eta(s, D_\chi^\sharp(\sigma)) = \eta_0(s, D_\chi^\sharp(\sigma)) + \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty \operatorname{Tr}(\Pi_+ D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) t^{\frac{s-1}{2}} dt. \quad (5.27)$$

Proof. Let λ be an eigenvalue of $D_\chi^\sharp(\sigma)$ such that $\operatorname{Re}(\lambda^2) > 0$. The Gamma function is defined by

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt, \quad \operatorname{Re}(s) > 0.$$

We apply now the following change of variables $t \mapsto t' = \lambda^2 t$ to see

$$(\lambda^2)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\lambda^2 t} dt. \quad (5.28)$$

Changing variables and using the Cauchy theorem to deform the contour of intergation back to the original one, we get

$$(\lambda_k^2)^{-\frac{s+1}{2}} = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty e^{-\lambda_k^2 t} t^{\frac{s-1}{2}} dt.$$

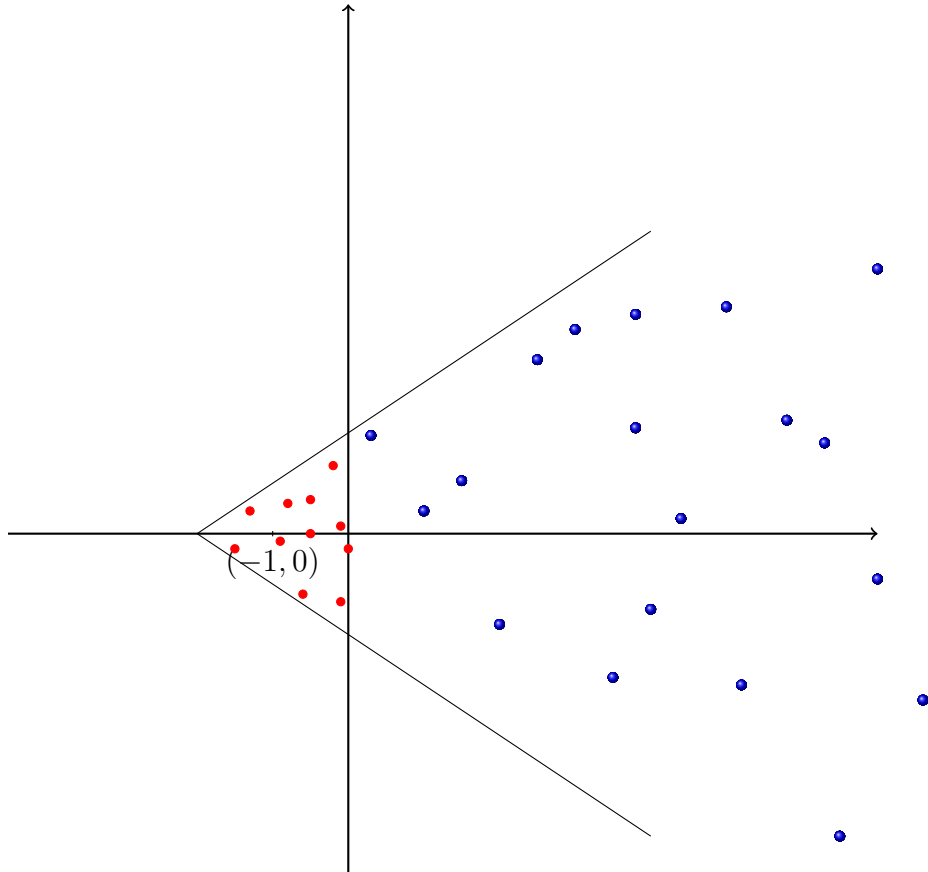


Figure 5.1: Finitely many eigenvalues of $(D_\chi^\sharp(\sigma))^2$ with negative real part.

We mention here that we can use the Lidskii's theorem ([Sim05, Theorem 3.7, p.35]) to express the trace of the operator $D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2}$ in terms of its eigenvalues λ_k :

$$\mathrm{Tr}(D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2}) = \sum_{\lambda_k \neq 0} m_s(\lambda_k) \lambda_k e^{-t\lambda_k^2}.$$

Taking the sum over the eigenvalues λ_k of $D_\chi^\sharp(\sigma)$, counting also their algebraic multiplicities, we have

$$\mathrm{Tr}(\Pi_+ D_\chi^\sharp(\sigma) ((D_\chi^\sharp(\sigma))^2)^{-\frac{s+1}{2}}) = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty \mathrm{Tr}(\Pi_+ D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) t^{\frac{s-1}{2}} dt. \quad (5.29)$$

To prove the convergence of the above integral, we first observe that

$$\begin{aligned} \mathrm{Tr}(\Pi_+ D_\chi^\sharp(\sigma) ((D_\chi^\sharp(\sigma))^2)^{-\frac{s+1}{2}}) &= \int_0^1 \mathrm{Tr}(\Pi_+ D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) t^{\frac{s-1}{2}} dt + \\ &\int_1^\infty \mathrm{Tr}(\Pi_+ D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) t^{\frac{s-1}{2}} dt. \end{aligned} \quad (5.30)$$

Then, for the first integral in the right hand side of (5.30), we use the asymptotic expansion of the trace of the operator $D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2}$ (Appendix A, Lemma A.12.). We have

$$\begin{aligned} \int_0^1 \mathrm{Tr}(\Pi_+ D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) t^{\frac{s-1}{2}} dt &= \int_0^1 \dim V_\chi(a_0(x)t^{1/2} + O(t^{3/2})) t^{\frac{s-1}{2}} dt \\ &< \dim V_\chi a_0 \frac{4}{s+3}, \end{aligned} \quad (5.31)$$

which is a holomorphic function for $\mathrm{Re}(s) > 0$.

We continue with the second integral in the right hand side of (5.30). We set $c_0 := \frac{1}{2} \min\{\mathrm{Re}(\lambda_k^2) : \mathrm{Re}(\lambda_k^2) > 0, \lambda_k \neq 0\}$. Then,

$$\left| \sum_{\lambda_k \neq 0} \lambda_k e^{-t\lambda_k^2} \right| \leq c_1 e^{-\frac{t}{2}c_0}.$$

Therefore,

$$\int_1^\infty |\mathrm{Tr}(\Pi_+ D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) t^{\frac{s-1}{2}}| dt \leq c_1 \int_1^\infty e^{-\frac{t}{2}c_0} t^{\mathrm{Re}(s)-1} dt < \infty. \quad (5.32)$$

By equations (5.30), (5.31), and (5.32), it follows that the integral in the right hand side of (5.29) is well defined and hence

$$\eta(s, D_\chi^\sharp(\sigma)) = \eta_0(s, D_\chi^\sharp(\sigma)) + \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty \mathrm{Tr}(\Pi_+ D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) t^{\frac{s-1}{2}} dt.$$

□

CHAPTER 6

Meromorphic continuation of the zeta functions

6.1 Resolvent identities

This section is the heart of this thesis. We will provide the proof for the meromorphic continuation of the Selberg and Ruelle zeta functions to the whole complex plane \mathbb{C} . The main tool that we will use is the trace formulas for the operators $D_\chi^\sharp(\sigma)e^{-tD_\chi^\sharp(\sigma)^2}$, and $e^{-tA_\chi^\sharp(\sigma)}$, as well the generalized resolvent identity (Lemma 6.1 below). In addition, the logarithmic derivatives $L(s), L_S(s), L^s(s)$ (cf. Lemma 2.8 in Section 2.4) of the Selberg, super, and symmetrized zeta functions, respectively, occur in the proofs, since they are closely related to the contribution of the hyperbolic conjugacy classes in the trace formulas. By this relation, we will obtain the meromorphic continuation of the zeta functions.

Let A be a closed linear operator, defined on a dense subspace of $\mathcal{D}(A)$ of a Hilbert space \mathcal{H} . Let $a \in \mathbb{C} - \text{spec}(-A)$. We set $R(a) := (A + a \text{Id})^{-1} = (A + a)^{-1}$. Then, the resolvent identity states

$$R(a) - R(b) = (b - a)R(a)R(b),$$

for all $a, b \in \mathbb{C} - \text{spec}(-A)$. The generalized resolvent identity is described in the following Lemma.

Lemma 6.1. *Let $s_1, \dots, s_N \in \mathbb{C} - \text{spec}(-A)$, $N \in \mathbb{N}$, such that $s_i \neq s_j$ for all*

$i \neq j$. Then,

$$\prod_{i=1}^N R(s_i) = \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j - s_i} \right) R(s_i). \quad (6.1)$$

Proof. This is proved in [BO95, Lemma 3.5]. \square

We will use also the following lemmata.

Lemma 6.2. *Let $s_1, \dots, s_N \in \mathbb{C}$, $N \in \mathbb{N}$, such that $s_i \neq s_j$ for all $i \neq j$ and let $l = 0, 1, \dots, N-2$. Then, we have*

$$\sum_{i=1}^N s_i^{2l} \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) = 0. \quad (6.2)$$

Proof. This follows from [BO95, Lemma 3.6], applied to s_i^2 . \square

Lemma 6.3. *Let $s_1, \dots, s_N \in \mathbb{C}$, $N \in \mathbb{N}$, such that $s_i \neq s_j$ for all $i \neq j$. Then,*

$$\sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} = O(t^{N-1}), \quad (6.3)$$

as $t \rightarrow 0^+$.

Proof. We will use the Taylor expansion of the exponential function $e^{-ts_i^2}$. We have as $t \rightarrow 0^+$

$$\begin{aligned} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} &= \sum_{k=1}^{N-2} \sum_{i=1}^N \frac{(-t)^k}{k!} s_i^{2k} \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) + O(t^{N-1}) \\ &= \sum_{k=1}^{N-2} \frac{(-t)^k}{k!} \sum_{i=1}^N s_i^{2k} \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) + O(t^{N-1}) = O(t^{N-1}), \end{aligned}$$

where in the last equality we used Lemma 6.2. \square

Lemma 6.4. *Let $s_1 \in \mathbb{C}$ such that $\operatorname{Re}(s_i^2) > 0$ for all $i = 1, \dots, N$. Then, the following integral*

$$\int_0^\infty \int_{\mathbb{R}} \sum_{k=1}^N \left(\prod_{\substack{j=1 \\ j \neq k}}^N \frac{1}{s_j^2 - s_k^2} \right) e^{-t(s_k^2 + \lambda^2)} P_\sigma(i\lambda) d\lambda dt \quad (6.4)$$

converges absolutely.

Proof. We have as $t \rightarrow \infty$,

$$\int_{\mathbb{R}} \left| \sum_{k=1}^N \left(\prod_{\substack{j=1 \\ j \neq k}}^N \frac{1}{s_j^2 - s_k^2} \right) e^{-t(s_k^2 + \lambda^2)} P_{\sigma}(i\lambda) \right| d\lambda = O(e^{-t\epsilon}), \quad (6.5)$$

for some $\epsilon > 0$.

We use now the fact the $P(i\lambda)$ is an even polynomial of degree $2n$ ([Mia79, 264-265]). If we make a change of variables $\lambda' \mapsto \lambda\sqrt{t}$, we get as $t \rightarrow 0^+$,

$$\int_{\mathbb{R}} \left| e^{-t\lambda^2} P_{\sigma}(i\lambda) \right| d\lambda = O(t^{-d/2}). \quad (6.6)$$

Hence, if we combine (6.3) and (6.6) we have that as $t \rightarrow 0^+$,

$$\int_{\mathbb{R}} \left| \sum_{k=1}^N \left(\prod_{\substack{j=1 \\ j \neq k}}^N \frac{1}{s_j^2 - s_k^2} \right) e^{-t(s_k^2 + \lambda^2)} P_{\sigma}(i\lambda) \right| d\lambda = O(t^{-d/2+N-1}). \quad (6.7)$$

The assertion follows from (6.5) and (6.7). \square

6.2 Meromorphic continuation of the super zeta function

Let $N \in \mathbb{N}$. Let $s_i, i = 1, \dots, N$ be complex numbers such that $s_i \in \mathbb{C} - \text{spec}(-D_{\chi}^{\sharp}(\sigma)^2)$. We consider the resolvent operator

$$R(s_i^2) = (D_{\chi}^{\sharp}(\sigma)^2 + s_i^2)^{-1}.$$

We want to obtain the trace class property of the operators

$$\prod_{i=1}^N R(s_i^2) \\ D_{\chi}^{\sharp}(\sigma) \prod_{i=1}^N R(s_i^2).$$

In order to obtain the trace class property of these operators, we take sufficient large $N \in \mathbb{N}$, such that

- for $N > \frac{d}{2}$,

$$\text{Tr} \left(\prod_{i=1}^N R(s_i^2) \right) < \infty. \quad (6.8)$$

- for $N > \frac{d}{2} + 1$,

$$\mathrm{Tr}(D_\chi^\sharp(\sigma) \prod_{i=1}^N R(s_i^2)) < \infty. \quad (6.9)$$

We denote the space of pseudodifferential operators of order k by ψDO^k . To prove the trace class property of the operators above, we observe at first that $\prod_{i=1}^N R(s_i^2) \in \psi DO^{-2N}$.

Let Δ be the Bochner-Laplace operator with respect to some metric, acting on $C^\infty(X, E_{\tau_s(\sigma)} \otimes E_\chi)$. Then, Δ is a second-order elliptic differential operator, which is formally self-adjoint and non-negative, i.e. $\Delta \geq 0$. Then, by Weyl's law, we have that for $N > \frac{d}{2}$,

$$(\Delta + \mathrm{Id})^{-N}$$

is a trace class operator.

Moreover,

$$B := (\Delta + \mathrm{Id})^N \prod_{i=1}^N R(s_i^2)$$

is ψDO of order zero. Hence, it defines a bounded operator in $L^2(X, E_{\tau_s(\sigma)} \otimes E_\chi)$. Thus,

$$\prod_{i=1}^N R(s_i^2) = (\Delta + \mathrm{Id})^{-N} B$$

is a trace class operator.

We recall here the following expressions of the resolvents. Let $s_1, \dots, s_N \in \mathbb{C}$ such that $\mathrm{Re}(s_i^2) > -c$, for all $i = 1, \dots, N$, where c is a real number such that $\mathrm{spec}(D_\chi^\sharp(\sigma)^2) \subset \{z \in \mathbb{C} : \mathrm{Re}(z) > c\}$.

Then,

$$D_\chi^\sharp(\sigma)(D_\chi^\sharp(\sigma)^2 + s_i^2)^{-1} = \int_0^\infty e^{-ts_i^2} D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2} dt \quad (6.10)$$

$$(A_\chi^\sharp(\sigma) + s_i^2)^{-1} = \int_0^\infty e^{-ts_i^2} e^{-tA_\chi^\sharp(\sigma)} dt. \quad (6.11)$$

Proposition 6.5. *Let $N \in \mathbb{N}$ with $N > d/2 + 1$. Let $s_1, \dots, s_N \in \mathbb{C}$ with $s_i \neq s_j$ for all $i \neq j$ such that $\mathrm{Re}(s_i^2) > -c$, for all $i = 1, \dots, N$, where c is a real number such that $\mathrm{spec}(D_\chi^\sharp(\sigma)^2) \subset \{z \in \mathbb{C} : \mathrm{Re}(z) > c\}$. Let $L^s(s) := \frac{d}{ds} \log(Z^s(s; \sigma, \chi))$ be the logarithmic derivative of the super zeta function. Then,*

$$\mathrm{Tr}(D_\chi^\sharp(\sigma) \prod_{i=1}^N (D_\chi^\sharp(\sigma)^2 + s_i^2)^{-1}) = -\frac{i}{2} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) L^s(s_i). \quad (6.12)$$

Proof. By Lemma 6.1 and formula (6.10), we have

$$D_\chi^\sharp(\sigma) \prod_{i=1}^N (D_\chi^\sharp(\sigma)^2 + s_i^2)^{-1} = \int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2} dt.$$

The operators $D_\chi^\sharp(\sigma) \prod_{i=1}^N (D_\chi^\sharp(\sigma)^2 + s_i^2)^{-1}$, and $D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2}$ are both of trace class. Then,

$$\begin{aligned} D_\chi^\sharp(\sigma) \prod_{i=1}^N (D_\chi^\sharp(\sigma)^2 + s_i^2)^{-1} &= \int_\epsilon^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2} dt \\ &\xrightarrow[\epsilon \rightarrow 0]{R \rightarrow \infty} \int_\epsilon^R \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2} dt, \end{aligned}$$

where the limit is taken with respect to the trace norm $\|\mathcal{A}\|_1 := \text{Tr}|\mathcal{A}|$, with $\mathcal{A} = D_\chi^\sharp(\sigma) \prod_{i=1}^N (D_\chi^\sharp(\sigma)^2 + s_i^2)^{-1}$, or $D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2}$. We have

$$\begin{aligned} \text{Tr} \left(\int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2} dt \right) &= \\ \text{Tr} \left(\lim_{\substack{\epsilon \rightarrow 0 \\ R \rightarrow \infty}} \int_\epsilon^R \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2} dt \right). \end{aligned}$$

But,

$$\begin{aligned} \text{Tr} \left(\int_\epsilon^R \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2} dt \right) &= \\ \int_\epsilon^R \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \text{Tr}(D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2}) dt. \end{aligned}$$

Hence, it is sufficient to show that the limit

$$\lim_{\substack{\epsilon \rightarrow 0 \\ R \rightarrow \infty}} \int_\epsilon^R \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \text{Tr}(D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2}) dt$$

exists. We study the behavior of the integral in the equation above as $\epsilon \rightarrow 0$. By Lemma 6.3 we have that as $t \rightarrow 0^+$

$$\sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} = O(t^{N-1}).$$

Also, by [GS95, Theorem 2.7, p.503-504], there exists a short-time asymptotic expansion of the kernel of the operator $D_\chi^\sharp(\sigma)e^{-tD_\chi^\sharp(\sigma)^2}$

$$\mathrm{Tr}(D_\chi^\sharp(\sigma)e^{-tD_\chi^\sharp(\sigma)^2}) \sim_{t \rightarrow 0^+} t^{-d/2}.$$

We have that as $t \rightarrow 0^+$

$$\left| \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \mathrm{Tr}(D_\chi^\sharp(\sigma)e^{-tD_\chi^\sharp(\sigma)^2}) \right| \leq Ct,$$

where C is a positive constant. All in all, we have proved

$$\mathrm{Tr}(D_\chi^\sharp(\sigma) \prod_{i=1}^N (D_\chi^\sharp(\sigma)^2 + s_i^2)^{-1}) = \int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \mathrm{Tr}(D_\chi^\sharp(\sigma)e^{-tD_\chi^\sharp(\sigma)^2}) dt.$$

We apply now the trace formula (5.24) for the operator $D_\chi^\sharp(\sigma)e^{-tD_\chi^\sharp(\sigma)^2}$. Then, we get

$$\begin{aligned} \mathrm{Tr}(D_\chi^\sharp(\sigma) \prod_{i=1}^N (D_\chi^\sharp(\sigma)^2 + s_i^2)^{-1}) &= \int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \\ &\quad \left\{ \sum_{[\gamma] \neq e} \frac{-2\pi i}{(4\pi t)^{3/2}} \frac{l^2(\gamma) \mathrm{tr}(\chi(\gamma) \otimes (\sigma(m_\gamma) - w\sigma(m_\gamma)))}{n_\Gamma(\gamma) D(\gamma)} e^{-l^2(\gamma)/4t} \right\}. \end{aligned} \tag{6.13}$$

If we use the formula (cf. [EMOT54, p.146, (28)])

$$\int_0^\infty e^{-ts^2} \frac{1}{(4\pi t)^{3/2}} e^{-l^2(\gamma)/4t} dt = \frac{e^{-l(\gamma)s}}{4\pi l(\gamma)}$$

equation (6.13) becomes

$$\begin{aligned} \mathrm{Tr}(D_\chi^\sharp(\sigma) \prod_{i=1}^N (D_\chi^\sharp(\sigma)^2 + s_i^2)^{-1}) &= \frac{-i}{2} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \\ &\quad \left\{ \sum_{[\gamma] \neq e} \frac{l(\gamma) \mathrm{tr}(\chi(\gamma) \otimes (\sigma(m_\gamma) - w\sigma(m_\gamma)))}{n_\Gamma(\gamma) D(\gamma)} e^{-l(\gamma)s_i} \right\}. \end{aligned}$$

Hence, by equation (2.26) we get

$$\mathrm{Tr}(D_\chi^\sharp(\sigma) \prod_{i=1}^N (D_\chi^\sharp(\sigma)^2 + s_i^2)^{-1}) = \frac{-i}{2} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) L^s(s_i).$$

□

The meromorphic continuation of the super zeta function follows from the Proposition (6.5) above.

Theorem 6.6. *The super zeta function $Z^s(s; \sigma, \chi)$ admits a meromorphic continuation to the whole complex plane \mathbb{C} . The singularities are located at $\{s_k^\pm = \pm i\lambda_k : \lambda_k \in \mathrm{spec}(D_\chi^\sharp(\sigma)), k \in \mathbb{N}\}$ of order $\pm m_s(\lambda_k)$, where $m_s(\lambda_k) = m(\lambda_k) - m(-\lambda_k) \in \mathbb{N}$ and $m(\pm\lambda_k)$ denotes the algebraic multiplicity of the eigenvalue $\pm\lambda_k$.*

Proof. We define the function $\Phi(s_1, s_2, \dots, s_N)$ of the complex variables s_1, s_2, \dots, s_N by

$$\Phi(s_1, s_2, \dots, s_N) = -\frac{i}{2} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) L^s(s_i). \quad (6.14)$$

By Lidskii's theorem and Lemma 6.1 the left side of (6.12) is equal to

$$\sum_{\lambda_k} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) m_s(\lambda_k) \lambda_k \frac{1}{(\lambda_k)^2 + s_i^2} = \Phi(s_1, s_2, \dots, s_N). \quad (6.15)$$

We fix the complex numbers $s_i, i = 2, \dots, N$ with $s_i \neq s_j$ for $i, j = 2, \dots, N$ and let the complex number $s = s_1$ vary. Hence,

$$\Phi(s, s_2, \dots, s_N) = \Phi(s)$$

The term that contains the logarithmic derivative $L^s(s)$ in $\Phi(s)$ is of the form

$$\frac{-i}{2} \left(\prod_{j=2}^N \frac{1}{s_j^2 - s^2} \right) L^s(s). \quad (6.16)$$

The term of

$$\sum_{\lambda_k} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) m_s(\lambda_k) \lambda_k \frac{1}{(\lambda_k)^2 + s_i^2}$$

which is singular at $s = \pm i\lambda_k$, $k \in \mathbb{N}$ is

$$\left(\prod_{j=2}^N \frac{1}{s_j^2 - s^2} \right) m_s(\lambda_k) \lambda_k \frac{1}{(\lambda_k)^2 + s^2}.$$

If we multiply both sides of (6.15) by

$$2i \prod_{j=2}^N (s_j^2 - s^2),$$

we see that the residue of the logarithmic derivative $L^s(s)$ at $\pm i\lambda_k$ is $\pm m_s(\lambda_k)$.

By (2.26), $L^s(s)$ decreases exponentially as $\operatorname{Re}(s) \rightarrow \infty$. Hence, the integral

$$\int_s^\infty L^s(w) dw$$

over a path connecting s and infinity is well defined and

$$\log Z^s(s; \sigma, \chi) = - \int_s^\infty L^s(w) dw. \quad (6.17)$$

The integral above depends on the choice of the path, because $L^s(s)$ has singularities at s_k^\pm . Nevertheless, since all the residues of the singularities are integers, it follows that the exponential of the integral in the right hand side of (6.17) is independent of the choice of the path. The meromorphic continuation of the super zeta function $Z^s(s; \sigma, \chi)$ to the whole complex plane follows. \square

6.3 Meromorphic continuation of the Selberg zeta function

We study first the case (a). Let $N \in \mathbb{N}$ with $N > d/2$. Let $s_1, \dots, s_N \in \mathbb{C}$ with $s_i \neq s_j$ for all $i \neq j$ such that $\operatorname{Re}(s_i^2) > -C$, for all $i = 1, \dots, N$, where C is a real number such that $\operatorname{spec}(A_\chi^\sharp(\sigma)) \subset \{z \in \mathbb{C} : \operatorname{Re}(z) > C\}$.

By Lemma 6.1 and equation (6.11) we have

$$\prod_{i=1}^N (A_\chi^\sharp(\sigma) + s_i^2)^{-1} = \int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} e^{-tA_\chi^\sharp(\sigma)} dt.$$

Arguing as in the proof of Proposition 6.5, we can plug the trace in both sides of the equation above.

Then,

$$\mathrm{Tr} \prod_{i=1}^N (A_{\chi}^{\sharp}(\sigma) + s_i^2)^{-1} = \int_0^{\infty} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \mathrm{Tr} e^{-tA_{\chi}^{\sharp}(\sigma)} dt.$$

We insert now the trace formula (4.45) for the operator $A_{\chi}^{\sharp}(\sigma)$ and get

$$\begin{aligned} \mathrm{Tr} \prod_{i=1}^N (A_{\chi}^{\sharp}(\sigma) + s_i^2)^{-1} &= \int_0^{\infty} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \\ &\quad \left\{ \dim(V_{\chi}) \mathrm{Vol}(X) \int_{\mathbb{R}} e^{-t\lambda^2} P_{\sigma}(i\lambda) d\lambda + \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_{\Gamma}(\gamma)} L_{\mathrm{sym}}(\gamma; \sigma) \frac{e^{-l(\gamma)^2/(4t)}}{(4\pi t)^{1/2}} \right\} dt. \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{Tr} \prod_{i=1}^N (A_{\chi}^{\sharp}(\sigma) + s_i^2)^{-1} &= \dim(V_{\chi}) \mathrm{Vol}(X) \int_0^{\infty} \int_{\mathbb{R}} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} e^{-t\lambda^2} P_{\sigma}(i\lambda) d\lambda dt \\ &\quad + \sum_{i=1}^N \int_0^{\infty} \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \left\{ \sum_{[\gamma] \neq [e]} \frac{l(\gamma)}{n_{\Gamma}(\gamma)} L_{\mathrm{sym}}(\gamma; \sigma) \frac{e^{-l(\gamma)^2/(4t)}}{(4\pi t)^{1/2}} \right\} dt. \end{aligned} \tag{6.18}$$

The first sum in (6.18), which involves the double integral can be explicitly calculated. We set

$$I := \int_0^{\infty} \int_{\mathbb{R}} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} e^{-t\lambda^2} P_{\sigma}(i\lambda) d\lambda dt.$$

By Lemma 6.4, we can interchange the order of the integration and get

$$I = \int_{\mathbb{R}} \int_0^{\infty} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-t(s_i^2 + \lambda^2)} P_{\sigma}(i\lambda) dt d\lambda.$$

By [BO95, Lemma 3.5] and since P_σ is an even polynomial, we obtain the following convergent integral

$$I = \int_{\mathbb{R}} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \left(\frac{1}{\lambda^2 + s_i^2} \right) P_\sigma(i\lambda) d\lambda.$$

Using the Cauchy integral formula we have

$$I = \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{\pi}{s_i} P_\sigma(s_i). \quad (6.19)$$

For the second sum in (6.18) we use the formula (cf. [EMOT54, p.146,(27)])

$$\int_0^\infty e^{-ts^2} \frac{e^{-l(\gamma)^2/(4t)}}{(4\pi t)^{1/2}} dt = \frac{1}{2s} e^{-sl(\gamma)}. \quad (6.20)$$

Hence, equation (6.18) becomes by (6.19) and (6.20)

$$\begin{aligned} \operatorname{Tr} \prod_{i=1}^N (A_\chi^\sharp(\sigma) + s_i^2)^{-1} &= \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{\pi}{s_i} \dim(V_\chi) \operatorname{Vol}(X) P(s_i) \\ &+ \sum_{i=1}^N \frac{1}{2s_i} \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} L_{\text{sym}}(\gamma; \sigma) e^{-s_i l(\gamma)}. \end{aligned} \quad (6.21)$$

By (2.24) we have that the sum over the conjugacy classes $[\gamma]$ of Γ in the right hand side of (6.21) is equal to the logarithmic derivative $L(s_i) = \frac{d}{ds} \log(Z(s_i; \sigma, \chi))$ of the Selberg zeta function. Hence,

$$\begin{aligned} \operatorname{Tr} \prod_{i=1}^N (A_\chi^\sharp(\sigma) + s_i^2)^{-1} &= \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{\pi}{s_i} \dim(V_\chi) \operatorname{Vol}(X) P(s_i) \\ &+ \sum_{i=1}^N \frac{1}{2s_i} \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) L(s_i). \end{aligned} \quad (6.22)$$

We will use the equation above to prove the following theorem.

Theorem 6.7. *The Selberg zeta function $Z(s; \sigma, \chi)$ admits a meromorphic continuation to the whole complex plane \mathbb{C} . The set of the singularities equals $\{s_k^\pm = \pm i\sqrt{t_k} : t_k \in \operatorname{spec}(A_\chi^\sharp(\sigma)), k \in \mathbb{N}\}$. The orders of the singularities are equal to $m(t_k)$, where $m(t_k) \in \mathbb{N}$ denotes the algebraic multiplicity of the eigenvalue t_k . For $t_0 = 0$, the order of the singularity s_0 is equal to $2m(0)$.*

Proof. By (6.1) and (6.22) we get

$$\begin{aligned} \mathrm{Tr} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) (A_\chi^\#(\sigma) + s_i^2)^{-1} &= \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{\pi}{s_i} \dim(V_\chi) \mathrm{Vol}(X) P(s_i) \\ &\quad + \sum_{i=1}^N \frac{1}{2s_i} \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) L(s_i). \end{aligned}$$

Equivalently,

$$\begin{aligned} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} L(s_i) &= - \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{\pi}{s_i} \dim(V_\chi) \mathrm{Vol}(X) P(s_i) \\ &\quad + \sum_{\lambda_k} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{m(t_k)}{t_k + s_i^2}, \end{aligned} \tag{6.23}$$

If we multiply equation (6.23) by $2s_1$, we obtain

$$\begin{aligned} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{s_1}{s_i} L(s_i) &= - \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{2s_1}{s_i} \pi \dim(V_\chi) \mathrm{Vol}(X) P(s_i) \\ &\quad + \sum_{\lambda_k} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) 2s_1 \frac{m(t_k)}{t_k + s_i^2}. \end{aligned} \tag{6.24}$$

Let $\Psi(s_1, \dots, s_N)$ be the function of the complex numbers s_1, \dots, s_N , defined by

$$\Psi(s_1, \dots, s_N) := \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{s_1}{s_i} L(s_i).$$

We fix the complex numbers $s_i, i = 2, \dots, N$ with $s_i \neq s_j$ for $i, j = 2, \dots, N$ and let the complex number $s = s_1$ vary.

Put

$$\Psi(s, \dots, s_N) = \Psi(s).$$

Then, equation (6.24) becomes

$$\begin{aligned} \Psi(s) = & - \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{2s}{s_i} \pi \dim(V_\chi) \text{Vol}(X) P(s_i) \\ & + \sum_{\lambda_k} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) 2s \frac{m(t_k)}{t_k + s_i^2}, \end{aligned} \quad (6.25)$$

where $s_1 = s$. The term that contains the logarithmic derivative $L(s)$ in $\Psi(s)$ is of the form

$$\left(\prod_{j=2}^N \frac{1}{s_j^2 - s^2} \right) L(s). \quad (6.26)$$

The term of

$$\sum_{\lambda_k} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) 2s \frac{m(t_k)}{t_k + s_i^2},$$

which is singular at $\pm i\sqrt{t_k}$, $k \in \mathbb{N}$ is

$$\left(\prod_{j=2}^N \frac{1}{s_j^2 - s^2} \right) 2s \frac{m(t_k)}{t_k + s^2}.$$

We multiply both sides of the equality (6.25) by

$$\prod_{j=2}^N (s_j^2 - s^2).$$

Then, the residues of $L(s)$ at the points $\pm i\sqrt{t_k}$ are $m(t_k)$, for $k \neq 0$, and $2m(0)$ for $k = 0$.

By (2.24), $L(s)$ decreases exponentially as $\text{Re}(s) \rightarrow \infty$. Hence, the integral

$$\int_s^\infty L(w) dw$$

over a path connecting s and infinity is well defined and

$$\log Z(s; \sigma, \chi) = - \int_s^\infty L(w) dw. \quad (6.27)$$

The integral above depends on the choice of the path, because $L(s)$ has singularities. Since the residues of the singularities are integers, it follows as in the proof of Theorem 6.6 that the exponential of the integral in the right hand side of (6.27) is independent of the choice of the path. The meromorphic continuation of the Selberg zeta function $Z(s; \sigma, \chi)$ to the whole complex plane follows. \square

6.4 Meromorphic continuation of the symmetrized zeta function

We examine now the case (b). We use the same reasoning as in case (a). Let $N \in \mathbb{N}$ with $N > d/2$. We choose $s_1, \dots, s_N \in \mathbb{C}$ with $s_i \neq s_j$ for all $i \neq j$ such that $\operatorname{Re}(s_i^2) > -r$, for all $i = 1, \dots, N$, where r is a real number such that $\operatorname{spec}(A_\chi^\sharp(\sigma)) \subset \{z \in \mathbb{C} : \operatorname{Re}(z) > r\}$.

Then, by Lemma 6.1 and equation (6.11), we have

$$\prod_{i=1}^N (A_\chi^\sharp(\sigma) + s_i^2)^{-1} = \int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} e^{-tA_\chi^\sharp(\sigma)} dt.$$

As in the proof of Proposition 6.5, we can consider the trace of the operators in the formula above and get

$$\operatorname{Tr} \prod_{i=1}^N (A_\chi^\sharp(\sigma) + s_i^2)^{-1} = \int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \operatorname{Tr} e^{-tA_\chi^\sharp(\sigma)} dt.$$

We insert the trace formula (4.46) for the operator $e^{-tA_\chi^\sharp(\sigma)}$ and get

$$\begin{aligned} \operatorname{Tr} \prod_{i=1}^N (A_{\tau, \chi}^\sharp(\sigma) + s_i^2)^{-1} &= \\ &= \int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \left\{ 2 \dim(V_\chi) \operatorname{Vol}(X) \int_{\mathbb{R}} e^{-t\lambda^2} P_\sigma(i\lambda) d\lambda \right. \\ &\quad \left. + \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} L_{\text{sym}}(\gamma; \sigma + w\sigma) \frac{e^{-l(\gamma)^2/4t}}{(4\pi t)^{1/2}} \right\} dt. \end{aligned} \quad (6.28)$$

The first sum in the right hand side of (6.28) includes the double integral

$$I = \int_0^\infty \int_{\mathbb{R}} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} e^{-t\lambda^2} P_\sigma(i\lambda) d\lambda dt,$$

which has been computed in the Section 6.3. Equation (6.19) gives

$$I = \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{\pi}{s_i} P_\sigma(s_i).$$

Hence, equation (6.28) reads

$$\begin{aligned} \mathrm{Tr} \prod_{i=1}^N (A_\chi^\sharp(\sigma) + s_i^2)^{-1} &= \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{\pi}{s_i} 2 \dim(V_\chi) \mathrm{Vol}(X) P_\sigma(s_i) \\ &+ \sum_{i=1}^N \frac{1}{2s_i} \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \sum_{[\gamma] \neq [e]} \frac{l(\gamma)}{n_\Gamma(\gamma)} L_{\mathrm{sym}}(\gamma; \sigma + w\sigma) e^{-s_i l(\gamma)}. \end{aligned}$$

By (2.25) we can insert the logarithmic derivative $L_S(s)$ of the symmetrized zeta function. Then, we get

$$\begin{aligned} \mathrm{Tr} \prod_{i=1}^N (A_\chi^\sharp(\sigma) + s_i^2)^{-1} &= \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{\pi}{s_i} 2 \dim(V_\chi) \mathrm{Vol}(X) P_\sigma(s_i) \\ &+ \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} L_S(s_i). \end{aligned} \quad (6.29)$$

Equation (6.29) will give the meromorphic continuation of the symmetrized zeta function.

Theorem 6.8. *The symmetrized zeta function $S(s; \sigma, \chi)$ admits a meromorphic continuation to the whole complex plane \mathbb{C} . The set of the singularities equals $\{s_k^\pm = \pm i\sqrt{\mu_k} : \mu_k \in \mathrm{spec}(A_\chi^\sharp(\sigma)), k \in \mathbb{N}\}$. The orders of the singularities are equal to $m(\mu_k)$, where $m(\mu_k) \in \mathbb{N}$ denotes the algebraic multiplicity of the eigenvalue μ_k . For $\mu_0 = 0$, the order of the singularity s_0 is equal to $2m(0)$.*

Proof. Lemma (6.1) and equation (6.29) give

$$\begin{aligned} \sum_{\mu_k} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{m(\mu_k)}{\mu_k + s_i^2} &= \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{\pi}{s_i} 2 \dim(V_\chi) \mathrm{Vol}(X) P_\sigma(s_i) \\ &+ \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} L_S(s_i). \end{aligned}$$

We multiply the last equation by $2s_1$ get

$$\begin{aligned} \sum_{\mu_k} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) 2s_1 \frac{m(\mu_k)}{\mu_k + s_i^2} &= \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{4\pi s_1}{s_i} \dim(V_\chi) \text{Vol}(X) P_\sigma(s_i) \\ &+ \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{s_1}{s_i} L_S(s_i). \end{aligned} \quad (6.30)$$

We define the function $\Xi(s_1, \dots, s_N)$ of the complex variables s_1, \dots, s_N by

$$\Xi(s_1, \dots, s_N) := \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{s_1}{s_i} L_S(s_i).$$

We fix the complex numbers $s_i, i = 2, \dots, N$ with $s_i \neq s_j$ for $i, j = 2, \dots, N$ and let the complex number $s = s_1$ vary.

Then,

$$\Xi(s, \dots, s_N) = \Xi(s),$$

and equation (6.30) becomes

$$\begin{aligned} \sum_{\mu_k} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) 2s \frac{m(\mu_k)}{\mu_k + s_i^2} &= \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{4\pi s}{s_i} \dim(V_\chi) \text{Vol}(X) P_\sigma(s_i) \\ &+ \Xi(s). \end{aligned} \quad (6.31)$$

The term that contains the logarithmic derivative $L_S(s)$ in $\Xi(s)$ is of the form

$$\left(\prod_{j=2}^N \frac{1}{s_j^2 - s^2} \right) L_S(s). \quad (6.32)$$

The term of

$$\sum_{\mu_k} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) 2s \frac{m(\mu_k)}{\mu_k + s_i^2},$$

which is singular at $s = \pm i\sqrt{\mu_k}$, $k \in \mathbb{N}$ is

$$\left(\prod_{j=2}^N \frac{1}{s_j^2 - s^2} \right) 2s \frac{m(\mu_k)}{\mu_k + s^2}.$$

We multiply both sides of the equality (6.31) by

$$\prod_{j=2}^N (s_j^2 - s^2).$$

Then, the residues of $L_S(s)$ at the points $\pm i\sqrt{\mu_k}$ are $m(\mu_k)$, for $k \neq 0$ and $2m(0)$, for $k = 0$.

By (2.25), $L_S(s)$ decreases exponentially as $\operatorname{Re}(s) \rightarrow \infty$. Therefore, the integral

$$\int_s^\infty L_S(w)dw$$

over a path connecting s and infinity is well defined and

$$\log S(s; \sigma, \chi) = - \int_s^\infty L_S(w)dw. \quad (6.33)$$

The integral above depends on the choice of the path, because $L_S(s)$ has singularities. Since the residues of the singularities are integers, we can use the same argument as in the proof of Theorem 6.6. If we exponentiate the right hand side of (6.33), then this exponential is independent of the choice of the path. The meromorphic continuation of the symmetrized zeta function $S(s; \sigma, \chi)$ to the whole complex plane follows. □

Furthermore, we have the following theorem.

Theorem 6.9. *The Selberg zeta function $Z(s; \sigma, \chi)$ admits a meromorphic continuation to the whole complex plane \mathbb{C} . The set of the singularities equals to $\{s_k^\pm = \pm i\lambda_k : \lambda_k \in \operatorname{spec}(D_\chi^\sharp(\sigma)), k \in \mathbb{N}\}$. The orders of the singularities are equal to $\frac{1}{2}(\pm m_s(\lambda_k) + m(\lambda_k^2))$. For $\lambda_0 = 0$, the order of the singularity is equal to $m(0)$.*

Proof. We observe at first that

$$Z(s; \sigma, \chi) = \sqrt{S(s; \sigma, \chi)Z^s(s; \sigma, \chi)}.$$

Recall that by equation (5.8) we have $A_\chi^\sharp(\sigma) = (D_\chi^\sharp(\sigma))^2$. Hence, we can identify the eigenvalues μ_k of $A_\chi^\sharp(\sigma)$ with λ_k^2 , where $\lambda_k \in \operatorname{spec}(D_\chi^\sharp(\sigma))$. By Theorem 6.6 and Theorem 6.8, the product $S(s; \sigma, \chi)Z^s(s; \sigma, \chi)$ has its singularities at $s_k^\pm = \pm i\lambda_k$, of order $\pm m_s(\lambda_k) + m(\lambda_k^2)$. We need to prove that the order of the singularities of $Z(s; \sigma, \chi)$ is an even integer. This follows from the definition of the algebraic multiplicities $m_s(\lambda_k)$, $m(\lambda_k^2)$ and the construction of the locally homogenous vector bundles $E(\sigma)$, $E_{\tau_s(\sigma)}$ associated to the representations $\tau(\equiv \tau_\sigma)$, $\tau_s(\sigma)$ of K . By

Proposition 4.6 together with equation (4.27), and Proposition 5.1 together with equation (5.2), we can choose representations τ of K , such that $i^*(\tau) = \sigma + w\sigma$ and $E(\sigma) = E_{\tau_s(\sigma)}$ up to a \mathbb{Z}_2 -grading. Then, the twisted Bochner-Laplace operators $A_\chi^\sharp(\sigma)$ associated to the representations $\tau, \tau_s(\sigma)$, respectively, coincide. Hence, $m_s(\lambda_k) \equiv m(\lambda_k^2) \pmod{2}$. The assertion follows. \square

6.5 Meromorphic continuation of the Ruelle zeta function

In this section we will prove the meromorphic continuation of the Ruelle zeta function to the whole complex plane \mathbb{C} . In view of the previous results, in sections 6.3 and 6.4, we will use the meromorphic continuation of the Selberg zeta function. Furthermore, we will use Theorem 6.10 below, which provides a representation of the Ruelle zeta function as a product of Selberg zeta functions with shifted arguments.

Let the identification $\mathfrak{a}_\mathbb{C}^* \cong \mathbb{C} \ni \lambda$. Let $\alpha > 0$ be the unique positive root of the system $(\mathfrak{g}, \mathfrak{a})$. Let $\lambda: A \rightarrow \mathbb{C}^\times$ be the character, defined by $\lambda(a) = e^{\alpha(\log a)}$.

Let $\mathfrak{n}_\mathbb{C}$ be the complexification of the Lie algebra \mathfrak{n} . Let $\nu_p := \Lambda^p \text{Ad}_{\mathfrak{n}_\mathbb{C}}(MA)$ be the representation of MA in $\Lambda^p \mathfrak{n}_\mathbb{C}$ given by the p -th exterior power of the adjoint representation:

$$\nu_p := \Lambda^p \text{Ad}_{\mathfrak{n}_\mathbb{C}}: MA \rightarrow \text{GL}(\Lambda^p \mathfrak{n}_\mathbb{C}), \quad p = 0, 1, \dots, d-1.$$

For $p = 0, 1, \dots, d-1$, let $J_p \subset \{(\psi_p, \lambda): \psi_p \in \widehat{M}, \lambda \in \mathbb{C}\}$ be the subset consisting of all pairs of unitary irreducible representations of M and one dimensional representations of A such that, as MA -modules, the representations ν_p decompose as

$$\Lambda^p \mathfrak{n}_\mathbb{C} = \bigoplus_{(\psi_p, \lambda) \in J_p} V_{\psi_p} \otimes \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda \cong \widehat{\mathbb{C}}$ denotes the representation space of λ .

For $\sigma \in \widehat{M}$ we define

$$Z_p(s; \sigma, \chi) := \prod_{(\psi_p, \lambda) \in J_p} Z(s + \rho - \lambda; \psi_p \otimes \sigma, \chi). \quad (6.34)$$

We have then the following theorem, which gives a representation of $R(s; \sigma, \chi)$ as a product of $Z_p(s; \sigma, \chi)$ over p .

Theorem 6.10. *Let $\sigma \in \widehat{M}$. Then the Ruelle zeta function has the representation*

$$R(s; \sigma, \chi) = \prod_{p=0}^{d-1} Z_p(s; \sigma, \chi)^{(-1)^p}. \quad (6.35)$$

Proof. By (2.13) we have

$$\log Z(s + \rho - \lambda; \psi_p \otimes \sigma, \chi) = - \sum_{[\gamma] \neq e} \frac{1}{n_\Gamma(\gamma)} \operatorname{tr}(\chi(\gamma) \otimes \sigma(m_\gamma)) \frac{e^{(-s-2\rho+\lambda)l(\gamma)} \operatorname{tr}(\psi_p(m))}{\det(1 - \operatorname{Ad}(m_\gamma a_\gamma)_{\bar{\mathfrak{n}}})}. \quad (6.36)$$

We use now the fact that

$$\det(1 - \operatorname{Ad}(m_\gamma a_\gamma)_{\bar{\mathfrak{n}}}) = (-1)^{d-1} a_\gamma^{-2\rho} \det(1 - \operatorname{Ad}(m_\gamma a_\gamma)_{\mathfrak{n}}). \quad (6.37)$$

Hence if we insert (6.37) in (6.36), we get

$$\log Z(s + \rho - \lambda; \psi_p \otimes \sigma, \chi) = (-1)^d \sum_{[\gamma] \neq e} \frac{1}{n_\Gamma(\gamma)} \operatorname{tr}(\chi(\gamma) \otimes \sigma(m_\gamma)) \frac{e^{-sl(\gamma)} e^{\lambda l(\gamma)} \operatorname{tr}(\psi_p(m))}{\det(1 - \operatorname{Ad}(m_\gamma a_\gamma)_{\mathfrak{n}})}. \quad (6.38)$$

We have

$$\begin{aligned} \log \prod_{p=0}^{d-1} Z_p(s; \sigma, \chi)^{(-1)^p} &= \sum_{p=0}^{d-1} \log Z_p(s; \sigma, \chi)^{(-1)^p} \\ &= \sum_{p=0}^{d-1} (-1)^p \log Z_p(s; \sigma, \chi) \\ &= \sum_{p=0}^{d-1} (-1)^p \log \prod_{(\psi_p, \lambda) \in J_p} Z(s + \rho - \lambda; \psi_p \otimes \sigma, \chi) \\ &= \sum_{p=0}^{d-1} (-1)^p \sum_{(\psi_p, \lambda) \in J_p} \log \left(Z(s + \rho - \lambda; \psi_p \otimes \sigma, \chi) \right) \\ &= \sum_{p=0}^{d-1} (-1)^p \left((-1)^d \sum_{[\gamma] \neq e} \frac{1}{n_\Gamma(\gamma)} \operatorname{tr}(\chi(\gamma) \otimes \sigma(m_\gamma)) e^{-sl(\gamma)} \right. \\ &\quad \left. \frac{\sum_{(\psi_p, \lambda) \in J_p} e^{\lambda l(\gamma)} \operatorname{tr}(\psi_p(m))}{\det(1 - \operatorname{Ad}(m_\gamma a_\gamma)_{\mathfrak{n}})} \right), \end{aligned} \quad (6.39)$$

where in the third line we used the definition (6.34), and in the last line we used equation (6.38).

We recall now that for any endomorphism of a finite dimensional vector space W , we have

$$\det(\operatorname{Id}_W - W) = \sum_{p=0}^{\infty} (-1)^p \operatorname{tr}(\Lambda^p W).$$

If we apply this identity to $\text{Ad}(m_\gamma a_\gamma)_\mathfrak{n}$ we get

$$\sum_{p=0}^{d-1} (-1)^p \frac{\sum_{(\psi_p, \lambda) \in J_p} e^{\lambda l(\gamma)} \text{tr}(\psi_p(m))}{\det(1 - \text{Ad}(m_\gamma a_\gamma)_\mathfrak{n})} = 1.$$

Hence, by (6.39) we have

$$\begin{aligned} \log \prod_{p=0}^{d-1} Z_p(s; \sigma, \chi)^{(-1)^p} &= (-1)^d \sum_{[\gamma] \neq e} \frac{1}{n_\Gamma(\gamma)} \text{tr}(\chi(\gamma) \otimes \sigma(m_\gamma)) e^{-sl(\gamma)} \\ &= \log R(s; \sigma, \chi), \end{aligned}$$

by equation (2.19) in the proof of Proposition 2.5. \square

Theorem 6.11. *For every $\sigma \in \widehat{M}$, the Ruelle zeta function $R(s; \sigma, \chi)$ admits a meromorphic continuation to the whole complex plane \mathbb{C} .*

Proof. • **case(a)** The assertion follows from Theorem 6.7 together with Theorem 6.10.

• **case(b)** The assertion follows from Theorem 6.9 together with Theorem 6.10. \square

CHAPTER 7

The functional equations

After proving the meromorphic continuation of the Selberg and Ruelle zeta function in Chapter 6, we continue with providing functional equations for them. We will derive functional equations for the Selberg zeta function in case (a) and (b) (Theorem 7.2 and 7.6, respectively), the symmetrized zeta function (Theorem 7.4), and the super zeta function (Theorem 7.5). Furthermore, we will use these functional equations for proving the functional equations for the Ruelle and super Ruelle zeta function (Theorem 7.9 and 7.10, respectively).

7.1 Functional equations for the Selberg zeta function

We consider at first case (a). We use the formulas from the previous chapter, which were derived from the generalized resolvent identity. Hence, we consider the statement below as a corollary of Theorem 6.7.

Corollary 7.1. *The logarithmic derivative of the Selberg zeta function satisfies the following functional equation*

$$L(s) + L(-s) = -4\pi \dim(V_\chi) \text{Vol}(X) P_\sigma(s). \quad (7.1)$$

Proof. We recall here equation (6.23)

$$\begin{aligned} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} L(s_i) &= - \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{\pi}{s_i} \dim(V_\chi) \operatorname{Vol}(X) P_\sigma(s_i) \\ &\quad + \sum_{\lambda_k} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{m(t_k)}{t_k + s_i^2}, \end{aligned}$$

We fix again the complex numbers $s_2, \dots, s_N \in \mathbb{C}$ and we let $s_1 = s \in \mathbb{C}$ vary. Then, we substitute $s \mapsto -s$. The resulting equation will differ from (6.23) in the terms for $i = 1$. i.e.

$$\left(\prod_{\substack{j=2 \\ j \neq i}}^N \frac{1}{s_j^2 - s^2} \right) \frac{1}{2s} L(s) \mapsto \left(\prod_{\substack{j=2 \\ j \neq i}}^N \frac{1}{s_j^2 - s} \right) \frac{1}{-2s} L(-s).$$

Also, since the Plancherel polynomial $P_\sigma(s)$ is an even polynomial of s ,

$$- \left(\prod_{\substack{j=2 \\ j \neq i}}^N \frac{1}{s_j^2 - s^2} \right) \frac{\pi}{s} \dim(V_\chi) \operatorname{Vol}(X) P_\sigma(s) \mapsto \left(\prod_{\substack{j=2 \\ j \neq i}}^N \frac{1}{s_j^2 - s^2} \right) \frac{\pi}{s} \dim(V_\chi) \operatorname{Vol}(X) P_\sigma(s).$$

We subtract the resulting equation from (6.23).

In an obvious way the sum that includes the term

$$\left(\prod_{\substack{j=2 \\ j \neq i}}^N \frac{1}{s_j^2 - s^2} \right) \frac{m(t_k)}{t_k + s^2}$$

will be canceled out, as well as the terms that include the fixed complex numbers s_2, \dots, s_N . Hence,

$$\left(\prod_{\substack{j=2 \\ j \neq i}}^N \frac{1}{s_j^2 - s^2} \right) \frac{1}{2s} (L(s) + L(-s)) = - \left(\prod_{\substack{j=2 \\ j \neq i}}^N \frac{1}{s_j^2 - s^2} \right) \frac{2\pi}{s} \dim(V_\chi) \operatorname{Vol}(X) P_\sigma(s).$$

We multiply the above equation by the function

$$2s \prod_{j=2}^N (s_j^2 - s^2).$$

Then we have

$$L(s) + L(-s) = -4\pi \dim(V_\chi) \text{Vol}(X) P_\sigma(s).$$

□

Theorem 7.2. *The Selberg zeta function $Z(s; \sigma, \chi)$ satisfies the following functional equation*

$$\frac{Z(s; \sigma, \chi)}{Z(-s; \sigma, \chi)} = \exp \left(-4\pi \dim(V_\chi) \text{Vol}(X) \int_0^s P_\sigma(r) dr \right). \quad (7.2)$$

Proof. We integrate once over s and we exponentiate equation (7.1). The assertion follows. □

We consider now the case (b) and we use the same method as in case (a). The following Corollary is a consequence of Theorem 6.8.

Corollary 7.3. *The logarithmic derivative $L_S(s)$ of the symmetrized zeta function $S(s; \sigma, \chi)$ satisfies the following functional equation*

$$L_S(s) + L_S(-s) = -8\pi \dim(V_\chi) \text{Vol}(X) P_\sigma(s). \quad (7.3)$$

Proof. We recall equation (6.29) from the previous chapter.

$$\begin{aligned} \text{Tr} \prod_{i=1}^N (A_\chi^\sharp(\sigma) + s_i^2)^{-1} &= \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{\pi}{s_i} 2 \dim(V_\chi) \text{Vol}(X) P_\sigma(s_i) \\ &\quad + \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} L_S(s_i). \end{aligned}$$

We consider (6.29) at $s \mapsto -s$. We subtract the resulting equation from (6.29). Using the same argument as in Corollary 7.1, we have

$$\left(\prod_{j=2}^N \frac{1}{s_j^2 - s^2} \right) \frac{1}{2s} \{L(s) + L(-s)\} = - \left(\prod_{j=2}^N \frac{1}{s_j^2 - s^2} \right) \frac{4\pi}{s} \dim(V_\chi) \text{Vol}(X) P_\sigma(s)$$

We multiply the above equation by the function

$$2s \prod_{j=2}^N (s_j^2 - s^2),$$

and we get formula (7.3). □

Theorem 7.4. *The symmetrized zeta function satisfies the following functional equation*

$$\frac{S(s; \sigma, \chi)}{S(-s, \sigma, \chi)} = \exp \left(-8\pi \dim(V_\chi) \text{Vol}(X) \int_0^s P_\sigma(r) dr \right). \quad (7.4)$$

Proof. We integrate over s and we exponentiate equation (7.3). The assertion follows. \square

Theorem 7.5. *The super zeta function satisfies the functional equation*

$$Z^s(s; \sigma, \chi) Z^s(-s; \sigma, \chi) = e^{2\pi i \eta(0, D_\chi^\sharp(\sigma))}, \quad (7.5)$$

where the $\eta(0, D_\chi^\sharp(\sigma))$ is the eta invariant associated to the Dirac operator $D_\chi^\sharp(\sigma)$. Furthermore,

$$Z^s(0; \sigma, \chi) = e^{\pi i \eta(0, D_\chi^\sharp(\sigma))}. \quad (7.6)$$

Proof. By Proposition 6.5 we have

$$\text{Tr}(D_\chi^\sharp(\sigma) \prod_{i=1}^N (D_\chi^\sharp(\sigma)^2 + s_i^2)^{-1}) = \frac{-i}{2} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) L^s(s_i).$$

Equivalently,

$$\begin{aligned} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) L^s(s_i) &= 2i \text{Tr}(D_\chi^\sharp(\sigma) \prod_{i=1}^N (D_\chi^\sharp(\sigma)^2 + s_i^2)^{-1}) \\ &= 2i \text{Tr} \int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2} dt, \end{aligned}$$

where we have employed the equation

$$(D_\chi^\sharp(\sigma)^2 + s^2)^{-1} = \int_0^\infty e^{-ts^2} e^{-t(D_\chi^\sharp(\sigma))^2} dt.$$

Using the same argument as in the proof of Proposition 6.5, we obtain

$$\sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) L^s(s_i) = 2i \int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \text{Tr}(D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) dt. \quad (7.7)$$

We fix now $s_2, \dots, s_N \in \mathbb{C}$ and let $s_1 = s \in \mathbb{C}$ vary. Then, as a function of s , the sum

$$\sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) L^s(s_i) \quad (7.8)$$

determines $L^s(s)$, up to an even polynomial of s . This polynomial arises from the finite product

$$\left(\prod_{j=2}^N \frac{1}{s_j^2 - s^2} \right).$$

We choose s_i such that $\operatorname{Re}(s_i) \rightarrow \infty$. Then, in the left-hand side of (7.7), $L^s(s_i)$ decreases exponentially. Similarly, in the right hand side of (7.7), we have the integrals that include the exponentials $e^{-ts_i^2}$:

$$\int_0^\infty e^{-ts_i^2} \operatorname{Tr}(D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) dt.$$

Note that each of these integrals is well defined, since as $t \rightarrow \infty$, $\operatorname{Tr}(D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2})$ and $e^{-ts_i^2}$ decay exponentially, and as $t \rightarrow 0^+$, we use the asymptotic expansion of the trace of the operator $D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}$. By Lemma A.12,

$$\operatorname{Tr}(D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) \sim_{t \rightarrow 0} \dim(V_\chi) (\alpha_0(x) t^{1/2} + O(t^{3/2})),$$

with

$$\alpha_0 = \int_X a_0(x) d\mu(x),$$

where $a_0(x)$ is a smooth local invariant, and $\mu(x)$ is the volume measure determined by the riemannian metric on X . Consequently, we can write the right hand side of (7.7) as the finite sum of the integrals

$$\int_0^\infty e^{-ts_i^2} \operatorname{Tr}(D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) dt,$$

multiplied by the finite product

$$\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2}, \quad i = 1, \dots, N.$$

If we choose now $s_i \in \mathbb{C}$ such that $\operatorname{Re}(s_i^2) \rightarrow \infty$, then the integrals that contain the term $e^{-ts_i^2}$ decay exponentially. Hence, we can remove the $\sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right)$ -structure and get

$$L^s(s) = 2i \int_0^\infty e^{-ts^2} \operatorname{Tr}(D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) dt.$$

Let Π_+ (resp. Π_-) be the projection on the span of the root spaces corresponding to eigenvalues λ of $D_\chi^\sharp(\sigma)$ with $\operatorname{Re}(\lambda^2) > 0$ (resp. $\operatorname{Re}(\lambda^2) \leq 0$). Recall that there are only finitely many eigenvalues of $D_\chi^\sharp(\sigma)$ such that $\operatorname{Re}(\lambda^2) \leq 0$ (cf. Figure 5.1). We write

$$\begin{aligned} L^s(s) &= 2i \int_0^\infty e^{-ts^2} \operatorname{Tr}(\Pi_+ D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) dt \\ &\quad + 2i \int_0^\infty e^{-ts^2} \operatorname{Tr}(\Pi_- D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) dt. \end{aligned} \quad (7.9)$$

We set

$$I_+ := \int_0^\infty e^{-ts^2} \operatorname{Tr}(\Pi_+ D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) dt \quad (7.10)$$

$$I_- := \int_0^\infty e^{-ts^2} \operatorname{Tr}(\Pi_- D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) dt. \quad (7.11)$$

Then, for the integral I_+ we have

$$\begin{aligned} \int_s^\infty I_+ &= \int_s^\infty \int_0^\infty e^{-tw^2} \operatorname{Tr}(\Pi_+ D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) dt dw \\ &= \int_0^\infty \int_s^\infty e^{-tw^2} \operatorname{Tr}(\Pi_+ D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) dw dt. \end{aligned}$$

If we make the change of variables $w \mapsto \frac{1}{\sqrt{t}}u$, we get

$$\int_s^\infty I_+ = \int_0^\infty \int_{\sqrt{ts}}^\infty \frac{1}{\sqrt{t}} e^{-u^2} \operatorname{Tr}(\Pi_+ D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) du dt.$$

We use now the error function

$$\Phi(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du.$$

It holds

$$\frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du = 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du = 1 - \Phi(x).$$

Hence,

$$\int_s^\infty I_+ = \int_0^\infty \frac{\sqrt{\pi}}{2\sqrt{t}} \left(1 - \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{ts}} e^{-u^2} \right) \text{Tr}(\Pi_+ D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) du dt, \quad (7.12)$$

and

$$\int_{-s}^\infty I_+ = \int_0^\infty \frac{\sqrt{\pi}}{2\sqrt{t}} \left(1 - \frac{2}{\sqrt{\pi}} \int_0^{-\sqrt{ts}} e^{-u^2} \right) \text{Tr}(\Pi_+ D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) du dt. \quad (7.13)$$

We add together (7.12) and (7.13) to get

$$\int_s^\infty I_+ + \int_{-s}^\infty I_+ = \int_0^\infty \frac{\sqrt{\pi}}{\sqrt{t}} \text{Tr}(\Pi_+ D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) dt. \quad (7.14)$$

We treat now the integral I_- . Since there are only finitely many eigenvalues of $D_\chi^\sharp(\sigma)$ with $\text{Re}(\lambda^2) \leq 0$, we can interchange the order of integration and write

$$\begin{aligned} \int_s^\infty I_- &= \int_s^\infty \int_0^\infty e^{-tw^2} \text{Tr}(\Pi_- D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) dt dw \\ &= \sum_{\substack{\lambda \\ \text{Re}(\lambda^2) \leq 0}} \lambda \int_s^\infty \int_0^\infty e^{-tw^2} e^{-t\lambda^2} dt dw \\ &= \sum_{\substack{\lambda \\ \text{Re}(\lambda^2) \leq 0}} \lambda \int_s^\infty \frac{1}{w^2 + \lambda^2} dw. \end{aligned} \quad (7.15)$$

Substituting at (7.15) $s \mapsto -s$ and adding the resulting equation to (7.15), we obtain

$$\int_s^\infty I_- + \int_{-s}^\infty I_- = \int_{\mathbb{R}} I_- = \sum_{\substack{\lambda \\ \text{Re}(\lambda^2) \leq 0}} \lambda^{-s} \int_{\mathbb{R}} \frac{1}{w^2 + \lambda^2} dw.$$

By change of variables $w \mapsto w' = w/\lambda$ we have

$$\int_s^\infty I_- + \int_{-s}^\infty I_- = \int_{\mathbb{R}} I_- = \sum_{\substack{\text{Re}(\lambda) > 0 \\ \text{Re}(\lambda^2) \leq 0}} \pi - \sum_{\substack{\text{Re}(\lambda) < 0 \\ \text{Re}(\lambda^2) \leq 0}} \pi.$$

The sums over λ in the equation above are finite, because we sum over λ with $\operatorname{Re}(\lambda^2) \leq 0$ and there are only finitely many eigenvalues such that $\operatorname{Re}(\lambda^2) \leq 0$.

We use now the definition of the function $\eta_0(0, D_\chi^\sharp(\sigma))$ from Section 5.3:

$$\eta_0(s, D_\chi^\sharp(\sigma)) := \sum_{\substack{\operatorname{Re}(\lambda) > 0 \\ \operatorname{Re}(\lambda^2) \leq 0}} \lambda^{-s} - \sum_{\substack{\operatorname{Re}(\lambda) < 0 \\ \operatorname{Re}(\lambda^2) \leq 0}} \lambda^{-s}.$$

Then,

$$\int_s^\infty I_- + \int_{-s}^\infty I_- = \pi \eta_0(0, D_\chi^\sharp(\sigma)). \quad (7.16)$$

We recall here equation (6.17)

$$\log Z^s(s; \sigma, \chi) = \int_s^\infty L^s(w) dw,$$

and equation (5.27)

$$\eta(s, D_\chi^\sharp(\sigma)) = \eta_0(s, D_\chi^\sharp(\sigma)) + \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty \operatorname{Tr}(\Pi_+ D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2}) t^{\frac{s-1}{2}} dt.$$

Hence, by (7.9), (7.14) and (7.16) we get

$$\begin{aligned} \log Z^s(s; \sigma, \chi) + \log Z^s(-s; \sigma, \chi) &= \int_s^\infty L^s(w) dw + \int_{-s}^\infty L^s(w) dw \\ &= 2i \int_s^\infty I_+ dw + 2i \int_{-s}^\infty I_+ dw \\ &\quad + 2i \int_s^\infty I_- dw + 2i \int_{-s}^\infty I_- dw \\ &= 2\pi i (\eta_1(0, D_\chi^\sharp(\sigma)) + \eta_0(0, D_\chi^\sharp(\sigma))) \\ &= 2\pi i \eta(0, D_\chi^\sharp(\sigma)). \end{aligned}$$

Equation (7.5) follows by exponentiation the equation above. We get equation (7.6) by substituting $s = 0$ in the equation above. \square

We prove now the functional equation for the Selberg zeta function in case (b).

Theorem 7.6. *The Selberg zeta function satisfies the following functional equation*

$$\frac{Z(s; \sigma, \chi)}{Z(-s; w\sigma, \chi)} = e^{\pi i \eta(0, D_\chi^\sharp(\sigma))} \exp \left(-4\pi \dim(V_\chi) \operatorname{Vol}(X) \int_0^s P_\sigma(r) dr \right). \quad (7.17)$$

Proof. By Definition 2.8 and 2.9 of the symmetrized and super zeta function, respectively, we have

$$\begin{aligned} \frac{Z(s; \sigma, \chi)}{Z(-s; w\sigma, \chi)} &= \sqrt{\frac{S(s; \sigma, \chi)Z^s(s; \sigma, \chi)}{S(-s; w\sigma, \chi)Z^s(-s; w\sigma, \chi)}} \\ &= \sqrt{\frac{S(s; \sigma, \chi)Z^s(s; \sigma, \chi)Z^s(-s; \sigma, \chi)}{S(-s; \sigma, \chi)}} \\ &= e^{\pi i \eta(0, D_\chi^\sharp(\sigma))} \exp\left(-4\pi \dim(V_\chi) \text{Vol}(X) \int_0^s P_\sigma(r) dr\right), \end{aligned}$$

where in the last equation we have employed Theorem 7.4 and Theorem 7.5. \square

7.2 Functional equations for the Ruelle zeta function

As in the previous section we will distinguish two cases. In case (a), we prove the functional equations for $R(s; \sigma, \chi)$, which relates its value at s with that at $-s$ (Theorem 7.9). In case (b), we prove functional equations for the super Ruelle zeta function, which is defined as follows.

Definition 7.7. Let $w \in W_A$, with $w \neq e_{W_A}$. We define the super Ruelle zeta function by

$$R^s(s; \sigma, \chi) := \frac{R(s; \sigma, \chi)}{R(s; w\sigma, \chi)}.$$

The functional equation relates also the value of $R^s(s; \sigma, \chi)$ at s with the one at $-s$ (Theorem 7.10). In addition, in case (b), the following equation holds (Theorem 7.10, equation (7.42))

$$\frac{R(s; \sigma, \chi)}{R(-s; w\sigma, \chi)} = e^{i\pi \eta(D^\sharp(\sigma \otimes \sigma_p))} \exp\left(-4\pi(d+1) \dim(V_\sigma) \dim(V_\chi) \text{Vol}(X)s\right).$$

We recall first from Section 6.5 the representation ν_p of MA in $\Lambda^p \mathfrak{n}_\mathbb{C}$, given by the p -th exterior power of the adjoint representation:

$$\nu_p := \Lambda^p \text{Ad}_{\mathfrak{n}_\mathbb{C}} : MA \rightarrow \text{GL}(\Lambda^p \mathfrak{n}_\mathbb{C}), \quad p = 0, 1, \dots, d-1.$$

For $p = 0, 1, \dots, d-1$, we consider $J_p \subset \{(\psi_p, \lambda) : \psi_p \in \widehat{M}, \lambda \in \mathbb{C}\}$ as the subset consisting of all pairs of unitary irreducible representations of M and one

dimensional representations of A such that, as MA -modules, the representations ν_p decompose as

$$\Lambda^p \mathfrak{n}_{\mathbb{C}} = \bigoplus_{(\psi_p, \lambda) \in J_p} V_{\psi_p} \otimes \mathbb{C}_{\lambda},$$

where $\mathbb{C}_{\lambda} \cong \mathbb{C}$ denotes the representation space of λ . By Poincaré duality we have for $p < \frac{d-1}{2}$,

$$J_{d-1-p} \subset \{(\psi_p, 2\rho - \lambda) : \psi_p \in \widehat{M}, \lambda \in \mathbb{C}\}. \quad (7.18)$$

We consider now the compact real forms G_d , and A_d of $G_{\mathbb{C}}$ and $A_{\mathbb{C}}$ respectively (cf. [Kna86, p.114]). Then $L := G_d/MA_d$ is Kähler manifold of dimension $\dim(L) = r$, which can be considered as the manifold of oriented geodesics of $X_d := G_d/K$.

For $\lambda \in \frac{1}{2}\mathbb{Z}$, we extend the one-dimensional representation of A to a representation of A_d . If $\lambda \in \mathbb{R}$ such that $\rho + \lambda \in \mathbb{Z}$, then the representation $\psi_p \otimes \lambda$ exists as a representation of MA_d . Let $E_{(\psi_p, \lambda)}$ be the holomorphic vector bundle over L , defined by

$$E_{(\psi_p, \lambda)} := G_d \times_{\psi_p \otimes \lambda} (V_{\psi_p} \otimes \mathbb{C}_{\lambda}) \rightarrow L.$$

Lemma 7.8. *Let $(\sigma, V_{\sigma}) \in \widehat{M}$. Let $P_{\psi_p \otimes \sigma}(s)$, $s \in \mathbb{C}$ be the Plancherel measure associated with the representation $\psi_p \otimes \sigma \in \widehat{M}$, $p = 0, \dots, d-1$. Let $f(s)$ be the polynomial of s given by*

$$\begin{aligned} f(s) &:= \frac{d}{ds} F(s) = (-1)^{\frac{d-1}{2}} P_{\psi_{\frac{d-1}{2}} \otimes \sigma}(s) + \\ &\quad \sum_{p=0}^{\frac{d-3}{2}} \sum_{(\psi_p, \lambda) \in J_p} (-1)^p [P_{\psi_p \otimes \sigma}(s + \rho - \lambda) + P_{\psi_p \otimes \sigma}(s - \rho + \lambda)] \\ &= \sum_{p=0}^{d-1} (-1)^p P_{\psi_p \otimes \sigma}(s; \rho, \lambda). \end{aligned} \quad (7.19)$$

Then,

$$f(s) = (d+1) \dim(V_{\sigma}). \quad (7.20)$$

Proof. Let Λ_M the highest weight of the ψ_p . Then, by the Borel-Weil-Bott Theorem (cf. [War72, Theorem 3.1.2.2]), we have that the representation space of the representation of G_d with highest weight Λ_M , can be realized as the space of the zero-Dolbeaux cohomology group $H^0(L, E_{(\psi_p, \lambda)})$ of $E_{(\psi_p, \lambda)}$. Moreover, all the higher cohomology groups $H^i(L, E_{(\psi_p, \lambda)})$ of $E_{(\psi_p, \lambda)}$ vanish for $i = 1, \dots, r$. Hence

$$\dim(H^0(L, E_{(\psi_p, \lambda)})) = \dim(V_{\psi_p} \otimes \mathbb{C}_{\lambda}). \quad (7.21)$$

On the other hand, by the Weyl's dimension formula (cf. [BO95, p. 47]) we have

$$\dim(V_{\psi_p} \otimes \mathbb{C}_{\lambda}) = P_{\psi_p}(\lambda + \rho). \quad (7.22)$$

By equations (7.21) and (7.22) we have

$$\begin{aligned}\chi(L, E_{(\psi_p, \lambda)}) &:= \sum_{q=0}^r (-1)^q \dim(H^q(L, E_{(\psi_p, \lambda)})) \\ &= \dim(H^0(L, E_{(\psi_p, \lambda)})) \\ &= P_{\psi_p}(\lambda + \rho).\end{aligned}$$

Therefore, for $\sigma \in \widehat{M}$,

$$P_{\psi_p \otimes \sigma}(\lambda + \rho) = \chi(L, E_{(\psi_p \otimes \sigma, \lambda)}). \quad (7.23)$$

Let $p < \frac{d-1}{2}$. Then, as MA_d -modules, the spaces $\{V_{\psi_p} \otimes V_{\sigma} \otimes \mathbb{C}_{s-\lambda} : (\psi_p, \lambda) \in J_p\}$ in the direct sum below decompose as

$$\begin{aligned}\bigoplus_{(\psi_p, \lambda) \in J_p} V_{\psi_p} \otimes V_{\sigma} \otimes \mathbb{C}_{s-\lambda} &= \bigoplus_{(\psi_p, \lambda) \in J_p} (V_{\psi_p} \otimes \mathbb{C}_{-\lambda}) \otimes (V_{\sigma} \otimes \mathbb{C}_s) \\ &= \Lambda^p \mathfrak{n}_{\mathbb{C}}^* \otimes (V_{\sigma} \otimes \mathbb{C}_s) \\ &= \Lambda^{p,0} T^* L \otimes (V_{\sigma} \otimes \mathbb{C}_s).\end{aligned} \quad (7.24)$$

Let $p > \frac{d-1}{2}$. Then, as MA_d -modules, the spaces $\{V_{\psi_p} \otimes V_{\sigma} \otimes \mathbb{C}_{s-2\rho+\lambda} : (\psi_p, \lambda) \in J_{p-1+d}\}$ decompose as

$$\begin{aligned}\bigoplus_{(\psi_p, \lambda) \in J_{p-1+d}} V_{\psi_p} \otimes V_{\sigma} \otimes \mathbb{C}_{s-2\rho+\lambda} &= \bigoplus_{(\psi_p, \lambda) \in J_{p-1+d}} (V_{\psi_p} \otimes \mathbb{C}_{\lambda}) \otimes (V_{\sigma} \otimes \mathbb{C}_{s-2\rho}) \\ &\quad \bigoplus_{(\psi_p, 2\rho-\lambda) \in J_{p-1+d}} (V_{\psi_p} \otimes \mathbb{C}_{2\rho-\lambda}) \otimes (V_{\sigma} \otimes \mathbb{C}_s) \\ &= \Lambda^p \mathfrak{n}_{\mathbb{C}}^* \otimes (V_{\sigma} \otimes \mathbb{C}_s) \\ &= \Lambda^{p,0} T^* L \otimes (V_{\sigma} \otimes \mathbb{C}_s).\end{aligned} \quad (7.25)$$

Therefore, by (7.19), (7.23), (7.24), and (7.25) we get

$$f(s) = \chi(L, \Lambda^{p,0} T^* L \otimes E_{\sigma, \lambda}). \quad (7.26)$$

We denote by $\mathcal{A}^{0,q}(L, E_{(\psi_p, \lambda)})$ the vector-valued $(0, q)$ -differential forms on L . Let the $\bar{\partial}$ -operator acting on $\mathcal{A}^{0,q}(L, E_{(\psi_p, \lambda)})$ and the Dirac-type operator

$$D_q := \bar{\partial} + \bar{\partial}^* : \bigoplus_{q=0}^{[r/2]} \mathcal{A}^{0,2q}(L, E_{(\psi_p, \lambda)}) \rightarrow \bigoplus_{q=0}^{[r/2]} \mathcal{A}^{0,2q+1}(L, E_{(\psi_p, \lambda)}). \quad (7.27)$$

Let \square_q be the complex Laplace operator, defined as follows.

$$\square_q := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} \circ \mathcal{A}^{0,q}(L, E_{(\psi_p, \lambda)}). \quad (7.28)$$

Then, by Hodge theory applied to \square_q , we have that there is an isomorphism of vector spaces $\mathcal{H}^{0,q}(L, E_{(\psi_p, \lambda)}) \cong H^q(L, E_{(\psi_p, \lambda)})$, where $\mathcal{H}^{0,q}(L, E_{(\psi_p, \lambda)}) := \ker(\square_q)$. Recall also the definition of the index of the Dirac-type operator $\bar{\partial} + \bar{\partial}^*$:

$$\text{ind}(\bar{\partial} + \bar{\partial}^*) := \dim \ker(\bar{\partial} + \bar{\partial}^*) - \dim \text{coker}(\bar{\partial} + \bar{\partial}^*).$$

We observe that

$$\begin{aligned} \chi(L, \Lambda^{p,0}T^*L \otimes E_{\sigma, \lambda}) &= \sum_{q=0}^r (-1)^q \dim(H^q(L, E_{(\psi_p, \lambda)})) \\ &= \sum_{q=0}^r (-1)^q \dim(\mathcal{H}^{0,q}(L, E_{(\psi_p, \lambda)})) \\ &= \sum_{q \text{ even}} \dim(\mathcal{H}^{0,q}(L, E_{(\psi_p, \lambda)})) - \sum_{q \text{ odd}} \dim(\mathcal{H}^{0,q}(L, E_{(\psi_p, \lambda)})) \\ &= \dim \ker(\bar{\partial} + \bar{\partial}^*) - \dim \ker(\bar{\partial} + \bar{\partial}^*)^* \\ &= \dim \ker(\bar{\partial} + \bar{\partial}^*) - \dim \text{coker}(\bar{\partial} + \bar{\partial}^*) \\ &= \text{ind}(\bar{\partial} + \bar{\partial}^*). \end{aligned} \quad (7.29)$$

We will use the index theorem for the operator $(\bar{\partial} + \bar{\partial}^*)$. By [GV92, Theorem 4.8], we have

$$\text{ind}(\bar{\partial} + \bar{\partial}^*) = \int_L \chi(TL) \wedge ch(E_{\sigma, \lambda}), \quad (7.30)$$

where $\chi(TL)$ denotes the Euler class of the tangent bundle of L , and $ch(E_{\sigma, \lambda})$ is the Chern character associated to $E_{\sigma, \lambda}$. Since $\chi(TL)$ is of top degree, then by the splitting principle for $E_{\sigma, \lambda}$ into line bundles, we have that $ch(E_{\sigma, \lambda})$ is a zero-form, and that

$$ch(E_{\sigma, \lambda}) \equiv ch_0(E_{\sigma, \lambda}) = \dim(E_{\sigma, \lambda}). \quad (7.31)$$

By [BT82, Proposition 11.24] we have that the Euler number for the Kähler manifold L equals its Euler characteristic

$$\int_L \chi(TL) = \chi(L). \quad (7.32)$$

By [Bot65, Theorem A], the Euler characteristic of L is equal to the order of the Weyl group $W(G_d, T)$, where T is a maximal torus subgroup of G_d . Then, if we consider the principle MA_d -fiber bundle G_d over L

$$G_d \rightarrow L = G_d/MA_d,$$

we get

$$\chi(G_d) = \chi(MA_d)\chi(L).$$

Hence, since $\chi(MA_d) = \text{order}(W(MA_d), T)$, we have

$$\chi(L) = \frac{\chi(G_d)}{\chi(MA_d)} = \frac{\text{order}(W(G_d, T))}{\text{order}(W(MA_d), T)}.$$

One can compute (cf. e.g. [Hei90, p.60]) that

$$\text{order}(W(\text{SO}(m))) = 2^{m-1}m!,$$

where $m \in \mathbb{N}$ is even. Therefore, we obtain

$$\chi(L) = \frac{2^{d-1/2}(d+1/2)!}{2^{d-3/2}(d-1/2)!} = d+1. \quad (7.33)$$

By equations (7.30), (7.31), (7.32) and (7.33) we have

$$\text{ind}(\bar{\partial} + \bar{\partial}^*) = (d+1) \dim(V_\sigma). \quad (7.34)$$

Hence, by (7.26), (7.29) and (7.34) we get

$$f(s) = (d+1) \dim(V_\sigma). \quad (7.35)$$

□

Theorem 7.9. *The Ruelle zeta function satisfies the following functional equation*

$$\frac{R(s; \sigma, \chi)}{R(-s; \sigma, \chi)} = \exp\left(-4\pi(d+1) \dim(V_\sigma) \dim(V_\chi) \text{Vol}(X)s\right). \quad (7.36)$$

Proof. By Proposition 6.10 and the definition of the zeta function $Z_p(s; \sigma, \chi)$ in Section 6.5. (equation 6.34) we have

$$R(s; \sigma, \chi) = \prod_{p=0}^{d-1} \left(\prod_{(\psi_p, \lambda) \in J_p} Z(s + \rho - \lambda; \psi_p \otimes \sigma, \chi) \right)^{(-1)^p}. \quad (7.37)$$

Then, equation (7.37) becomes by (7.18)

$$R(s; \sigma, \chi) = Z\left(s; \psi_{\frac{d-1}{2}} \otimes \sigma, \chi\right)^{(-1)^{\frac{d-1}{2}}} \prod_{p=0}^{\frac{d-3}{2}} \prod_{(\psi_p, \lambda) \in J_p} Z(s + \rho - \lambda; \psi_p \otimes \sigma, \chi) Z(s - \rho + \lambda; \psi_p \otimes \sigma, \chi)^{(-1)^p}.$$

Hence,

$$\frac{R(s; \sigma, \chi)}{R(-s; \sigma, \chi)} = \left(\frac{Z(s; \psi_{\frac{d-1}{2}} \otimes \sigma, \chi)}{Z(-s; \psi_{\frac{d-1}{2}} \otimes \sigma, \chi)} \right)^{(-1)^{\frac{d-1}{2}}} \prod_{p=0}^{\frac{d-3}{2}} \prod_{(\psi_p, \lambda) \in J_p} \left(\frac{Z(s + \rho - \lambda; \psi_p \otimes \sigma, \chi) Z(s - \rho + \lambda; \psi_p \otimes \sigma, \chi)}{Z(-s + \rho - \lambda; \psi_p \otimes \sigma, \chi) Z(-s - \rho + \lambda; \psi_p \otimes \sigma, \chi)} \right)^{(-1)^p}.$$

We will use the functional equations for the Selberg zeta function from Section 7.1.

By Theorem 7.2 (equation (7.2)), we get

$$\frac{R(s; \sigma, \chi)}{R(-s; \sigma, \chi)} = \exp \left\{ -4\pi \dim(V_\chi) \text{Vol}(X) \left((-1)^{\frac{d-1}{2}} \int_0^s P_{\psi_{\frac{d-1}{2}} \otimes \sigma}(r) dr + \sum_{p=0}^{\frac{d-3}{2}} \sum_{(\psi_p, \lambda) \in J_p} (-1)^p \left(\int_0^{s+\rho-\lambda} P_{\psi_p \otimes \sigma}(r) dr + \int_0^{s-\rho+\lambda} P_{\psi_p \otimes \sigma}(r) dr \right) \right) \right\}. \quad (7.38)$$

We set

$$F(s) = (-1)^{\frac{d-1}{2}} \int_0^s P_{\psi_{\frac{d-1}{2}} \otimes \sigma}(r) dr + \sum_{p=0}^{\frac{d-3}{2}} \sum_{(\psi_p, \lambda) \in J_p} (-1)^p \left(\int_0^{s+\rho-\lambda} P_{\psi_p \otimes \sigma}(r) dr + \int_0^{s-\rho+\lambda} P_{\psi_p \otimes \sigma}(r) dr \right).$$

Then,

$$\frac{d}{ds} F(s) = f(s),$$

where $f(s)$ as in (7.19). We can easily write from Lemma 7.8 above

$$\frac{R(s; \sigma, \chi)}{R(-s; \sigma, \chi)} = \exp \left(-4\pi \dim(V_\chi) \text{Vol}(X) [(d+1) \dim(V_\sigma) s + C] \right), \quad (7.39)$$

where $C \in \mathbb{R}$ is a real constant. On the other hand, if we set $s = 0$ in (7.39), we get $1 = \exp(-4\pi \dim(V_\chi) \text{Vol}(X) C)$, and hence $C = 0$. The assertion follows. \square

We exam now case (b). Let τ_p be the standard representation of K on $\Lambda^p \mathbb{R}^d \otimes \mathbb{C}$. Let (σ_p, V_{σ_p}) be the standard representation of M in $\Lambda^p \mathbb{R}^{d-1} \otimes \mathbb{C}$. We recall from

Section 6.5 the representation $\nu_p := \Lambda^p \text{Ad}_{\mathfrak{n}_{\mathbb{C}}}$ of MA on $\Lambda^p \mathfrak{n}_{\mathbb{C}}$ given by the p -th exterior power of the adjoint representation:

$$\nu_p := \Lambda^p \text{Ad}_{\mathfrak{n}_{\mathbb{C}}}: MA \rightarrow \text{GL}(\Lambda^p \mathfrak{n}_{\mathbb{C}}), \quad p = 0, 1, \dots, d-1.$$

Let $\alpha > 0$ be the unique positive root of the system $(\mathfrak{g}, \mathfrak{a})$. Let $\lambda_p: A \rightarrow \mathbb{C}^\times$ be the character, defined by $\lambda_p(a) = e^{p\alpha(\log a)}$. Then as a representation of MA one has $\nu_p = \sigma_p \otimes \lambda_p$.

We denote by $\mathbb{C}_p \cong \mathbb{C}$ the representation space of λ_p . Then, in the sense of MA -modules, we have

$$\Lambda^p \mathfrak{n}_{\mathbb{C}} = \Lambda^p \mathbb{R}^{d-1} \otimes \mathbb{C}_p. \quad (7.40)$$

Let $D_\chi^\sharp(\sigma)$ be the twisted Dirac operator acting on $C^\infty(X, E_{\tau_s(\sigma)} \otimes E_\chi)$. For our proposal, we define the twist $D_{p,\chi}^\sharp(\sigma)$ of the Dirac operator $D_\chi^\sharp(\sigma)$ acting on

$$\bigoplus_{p=0}^{d-1} C^\infty(X, E_{\tau_s(\sigma)} \otimes E_\chi \otimes (d-p)\Lambda^p T^*X).$$

The twisted Dirac operator $D_{p,\chi}^\sharp(\sigma)$ is defined in a similar way as the Dirac operator $D_\chi^\sharp(\sigma)$ in Chapter 5. We equip the bundle $\Lambda^p T^*X$ with the Levi-Civita connection of X , and we proceed as in Section 5.1.

Theorem 7.10. *The super Ruelle zeta function associated with a non-Weyl invariant representation $\sigma \in \widehat{M}$ satisfies the functional equation*

$$R^s(s; \sigma, \chi) R^s(-s; \sigma, \chi) = e^{2i\pi\eta(D_{p,\chi}^\sharp(\sigma))}, \quad (7.41)$$

where $\eta(D_{p,\chi}^\sharp(\sigma))$ denotes the eta invariant of the twisted Dirac operator $D_{p,\chi}^\sharp(\sigma)$. Moreover, the following equation holds

$$\frac{R(s; \sigma, \chi)}{R(-s; w\sigma, \chi)} = e^{i\pi\eta(D_{p,\chi}^\sharp(\sigma))} \exp\left(-4\pi(d+1) \dim(V_\sigma) \dim(V_\chi) \text{Vol}(X)s\right). \quad (7.42)$$

Proof. By [BO95, p. 23], we have

$$\begin{aligned} \sigma_p &= i^*((-1)^0 \tau_p + (-1)^1 \tau_{p-1} + \dots + (-1)^{p-1} (\tau_1 - \text{Id})), \quad p = 1, 2, \dots, d-1 \\ s^+ + s^- &= i^*(s), \quad \text{otherwise.} \end{aligned}$$

If we take the alternating sum of σ_p over p we get

$$\sum_{p=0}^{d-1} (-1)^p \sigma_p = i^* \left(\sum_{p=0}^{d-1} (-1)^p (d-p) \tau_p \right). \quad (7.43)$$

We write

$$\begin{aligned} R^s(s; \sigma, \chi)R^s(-s; \sigma, \chi) &= \frac{R(s; \sigma, \chi)}{R(s; w\sigma, \chi)} \frac{R(-s; \sigma, \chi)}{R(-s; w\sigma, \chi)} \\ &= \frac{R(s; \sigma, \chi)}{R(-s; w\sigma, \chi)} \frac{R(-s; \sigma, \chi)}{R(s; w\sigma, \chi)}. \end{aligned} \quad (7.44)$$

We will use now the representation (7.37) of the Ruelle zeta function. By the Poincaré duality we obtain

$$\begin{aligned} R(s; \sigma, \chi) &= Z(s; \sigma_{\frac{d-1}{2}} \otimes \sigma, \chi)^{(-1)^{\frac{d-1}{2}}} \\ &\quad \prod_{p=0}^{\frac{d-3}{2}} Z(s + \rho - \lambda; \sigma_p \otimes \sigma, \chi) Z(s - \rho + \lambda; \sigma_p \otimes \sigma, \chi)^{(-1)^p}. \end{aligned}$$

If we substitute the expression above in (7.44), we have

$$\begin{aligned} R^s(s; \sigma, \chi)R^s(-s; \sigma, \chi) &= \frac{Z(s; \sigma_{\frac{d-1}{2}} \otimes \sigma, \chi)^{(-1)^{\frac{d-1}{2}}}}{Z(-s; \sigma_{\frac{d-1}{2}} \otimes w\sigma, \chi)} \\ &\quad \cdot \frac{\prod_{p=0}^{\frac{d-3}{2}} (Z(s + \rho - \lambda; \sigma_p \otimes \sigma, \chi) Z(s - \rho + \lambda; \sigma_p \otimes \sigma, \chi))^{(-1)^p}}{\prod_{p=0}^{\frac{d-3}{2}} (Z(-s + \rho - \lambda; \sigma_p \otimes w\sigma, \chi) Z(-s - \rho + \lambda; \sigma_p \otimes w\sigma, \chi))^{(-1)^p}} \\ &\quad \cdot \frac{Z(-s; \sigma_{\frac{d-1}{2}} \otimes \sigma, \chi)^{(-1)^{\frac{d-1}{2}}}}{Z(s; \sigma_{\frac{d-1}{2}} \otimes w\sigma, \chi)} \\ &\quad \cdot \frac{\prod_{p=0}^{\frac{d-3}{2}} (Z(-s + \rho - \lambda; \sigma_p \otimes \sigma, \chi) Z(-s - \rho + \lambda; \sigma_p \otimes \sigma, \chi))^{(-1)^p}}{\prod_{p=0}^{\frac{d-3}{2}} (Z(s + \rho - \lambda; \sigma_p \otimes w\sigma, \chi) Z(s - \rho + \lambda; \sigma_p \otimes w\sigma, \chi))^{(-1)^p}}. \end{aligned}$$

By Theorem 7.5, we get

$$R^s(s; \sigma, \chi)R^s(-s; \sigma, \chi) = (e^{2i\pi\eta(0, D_\chi^\sharp(\sigma \otimes \sigma_{d-1/2}))})^{(-1)^{\frac{d-1}{2}}} \prod_{p=0}^{\frac{d-3}{2}} (e^{2i\pi\eta(0, D_\chi^\sharp(\sigma \otimes \sigma_p))})^{(-1)^p},$$

where we used the fact that the Plancherel polynomial is an even function. Here, $D_\chi^\sharp(\sigma \otimes \sigma_p)$ denotes the Dirac operator acting on the space

$$\bigoplus_{i=0}^p C^\infty(X, E_{\tau_s(\sigma)} \otimes E_\chi \otimes (p-i)\Lambda^i T^*X), \quad p = 0, 1, \dots, d-1. \quad (7.45)$$

Finally, we have

$$\begin{aligned} R^s(s; \sigma, \chi) R^s(-s; \sigma, \chi) &= e^{2i\pi \sum_{p=0}^{d-1} (-1)^p \eta(0, D_\chi^\sharp(\sigma \otimes \sigma_p))} \\ &= e^{2i\pi \eta(D_{p,\chi}^\sharp(\sigma))}, \end{aligned}$$

where $\eta(D_{p,\chi}^\sharp(\sigma))$ denotes the eta invariant of the operator $D_{p,\chi}^\sharp(\sigma)$, which by definition of $D_{p,\chi}^\sharp(\sigma)$ and equation (7.43), is given by

$$\eta(D_{p,\chi}^\sharp(\sigma)) = \sum_{p=0}^{d-1} (-1)^p \eta(0, D_\chi^\sharp(\sigma \otimes \sigma_p)).$$

For the functional equations (7.42) we have

$$\begin{aligned} \frac{R(s; \sigma, \chi)^2}{R(-s; w\sigma, \chi)^2} &= \frac{R(s; \sigma, \chi)}{R(-s; w\sigma, \chi)} \frac{R(-s; \sigma, \chi)}{R(s; w\sigma, \chi)} \frac{R(s; \sigma, \chi)}{R(-s; \sigma, \chi)} \frac{R(s; w\sigma, \chi)}{R(-s; w\sigma, \chi)} \\ &= e^{2i\pi \eta(D_{p,\chi}^\sharp(\sigma))} \frac{R(s; \sigma, \chi) R(s; w\sigma, \chi)}{R(-s; \sigma, \chi) R(-s; w\sigma, \chi)}, \end{aligned}$$

where we have employed the functional equation for the super Ruelle zeta function (7.41).

One can easily compute as in the proof of Theorem 7.9 (formula (7.39)) that

$$\frac{R(s; w\sigma, \chi)}{R(-s; w\sigma, \chi)} = \exp \left(-4\pi(d+1) \dim(V_\sigma) \dim(V_\chi) \text{Vol}(X)s \right).$$

Hence,

$$\frac{R(s; \sigma, \chi)^2}{R(-s; w\sigma, \chi)^2} = e^{2i\pi \eta(D_{p,\chi}^\sharp(\sigma))} \exp 2 \left(-4\pi(d+1) \dim(V_\sigma) \dim(V_\chi) \text{Vol}(X)s \right).$$

The assertion follows. □

CHAPTER 8

The determinant formula

We recall first from Appendix A (Lemma A.13) the asymptotic expansion of the trace of the operator $e^{-tA_\chi^\sharp(\sigma)}$:

$$\mathrm{Tr}(e^{-tA_\chi^\sharp(\sigma)}) \sim_{t \rightarrow 0^+} \dim(V_\chi) \sum_{j=0}^{\infty} a_j t^{j-\frac{d}{2}}. \quad (8.1)$$

Definition 8.1. The xi function associated to the operator $A_\chi^\sharp(\sigma)$ is defined by

$$\xi(z, s; \sigma) := \int_0^\infty e^{-ts^2} \mathrm{Tr}(e^{-tA_\chi^\sharp(\sigma)}) t^{z-1} dt, \quad (8.2)$$

for $\mathrm{Re}(s^2) > C$, where $C \in \mathbb{R}$ and $\mathrm{Re}(\lambda_i) > 0$, where $\lambda_i \in \mathrm{spec}(A_\chi^\sharp(\sigma))$.

Definition 8.2. We define the generalized zeta function $\zeta(z, s; \sigma)$ by

$$\zeta(z, s; \sigma) = \frac{1}{\Gamma(z)} \int_0^\infty e^{-ts^2} \mathrm{Tr}(e^{-tA_\chi^\sharp(\sigma)}) t^{z-1} dt, \quad (8.3)$$

for $\mathrm{Re}(s^2) > C$, where $C \in \mathbb{R}$ and $\mathrm{Re}(\lambda_i) > 0$, where $\lambda_i \in \mathrm{spec}(A_\chi^\sharp(\sigma))$.

The two functions converge absolutely and uniformly on compact subsets of the half-plane $\mathrm{Re}(z) > \frac{d}{2}$. Furthermore, they are differentiable in $s \in \mathbb{C}$.

Lemma 8.3. *The xi function $\xi(\cdot, s; \sigma)$ admits a meromorphic continuation to the whole complex plane \mathbb{C} . Furthermore, it has simple poles at $k_j = \frac{d}{2} - j$ with $\mathrm{res}(k_j, \xi(\cdot, s; \sigma)) = a_j$.*

Proof. We define the theta function $\theta(t)$ associated with the operator $e^{-tA_\chi^\sharp(\sigma)}$ by

$$\theta(t) := \operatorname{Tr}(e^{-tA_\chi^\sharp(\sigma)}) = \sum_{\lambda_j \in \operatorname{spec}(A_\chi^\sharp(\sigma))} m(\lambda_j) e^{-t\lambda_j},$$

where $m(\lambda_j)$ denotes the algebraic multiplicity of the eigenvalue λ_j . Then, $\xi(z, s; \sigma)$ defined by (8.2) is just the Mellin-Laplace transform of $\theta(t)$.

Since the spectrum of $A_\chi^\sharp(\sigma)$ is discrete and contained in a translate of a positive cone in \mathbb{C} (cf. Appendix A, Lemma A.11 and Figure A.1), there are only finitely many eigenvalues λ_j with $\operatorname{Re}(\lambda_j) \leq 0$.

For $N \in \mathbb{N}$ with $N > 1$, we have

$$\begin{aligned} \left| \sum_{j=1}^{\infty} m(\lambda_j) e^{-t\lambda_j} - \sum_{j=1}^N m(\lambda_j) e^{-t\lambda_j} \right| &= \left| \sum_{j=N+1}^{\infty} m(\lambda_j) e^{-t\lambda_j} \right| \\ &\leq \sum_{j=N+1}^{\infty} m(\lambda_j) e^{-t\operatorname{Re}(\lambda_j)}. \end{aligned} \quad (8.4)$$

We observe now that there are only finitely many eigenvalues λ_j such that $|\lambda_j| \leq c$, where c is positive constant. On the other hand, for every positive constant c there exists a positive integer N such that $\operatorname{Re}(\lambda_j) \geq c$, for every $j \geq N$. We consider an ordering $\operatorname{Re}(\lambda_{j_1}) \leq \operatorname{Re}(\lambda_{j_2}) \leq \operatorname{Re}(\lambda_{j_3}) \leq \dots$ of the real parts of the eigenvalues with $\operatorname{Re}(\lambda_j) \geq c$. Then for $t \geq 1$,

$$\begin{aligned} \sum_{j=N+1}^{\infty} m(\lambda_j) e^{-t\operatorname{Re}(\lambda_j)} &\leq \sum_{j=N+1}^{\infty} m(\lambda_j) e^{-t\operatorname{Re}(\lambda_j)/2} e^{-t\operatorname{Re}(\lambda_j)/2} \\ &\leq e^{-tc/2} \sum_{j=N+1}^{\infty} m(\lambda_j) e^{-t\operatorname{Re}(\lambda_j)/2} \\ &\leq e^{-tc/2} \sum_{j=N+1}^{\infty} m(\lambda_j) e^{-\operatorname{Re}(\lambda_j)/2}. \end{aligned} \quad (8.5)$$

To estimate the last sum, we will use the Weyl's law for the non self-adjoint operator $A_\chi^\sharp(\sigma)$. Given a positive constant c , we define the counting function $\mathcal{N}(c)$ by

$$\mathcal{N}(c) := \sum_{\substack{\lambda_j \in \operatorname{spec}(A_\chi^\sharp(\sigma)) \\ |\lambda_j| \leq c}} m(\lambda_j).$$

In [Mül11], the generalization of the Weyl's law for the non self-adjoint case is proved. By [Mül11, Lemma 2.2], we have

$$\mathcal{N}(c) = \frac{\operatorname{rank}(E(\sigma) \otimes E_\chi) \operatorname{Vol}(X)}{(4\pi)^{d/2} \Gamma(d/2 + 1)} c^{d/2} + o(c^{d/2}), \quad c \rightarrow \infty, \quad (8.6)$$

where $\text{rank}(E(\sigma) \otimes E_\chi)$ denotes the rank of the product vector bundle $E(\sigma) \otimes E_\chi$. To use the Weyl's law (8.6), we observe that for a real number $a > 1$ (the slope of the straight line of the cone, which all the eigenvalues λ_j of $A_\chi^\sharp(\sigma)$ are contained in), we have

$$\#\{j: |\text{Re}(\lambda_j)| \leq \lambda\} \leq \#\{j: |\lambda_j| \leq a\lambda\} \leq \mathcal{N}(a\lambda).$$

By (8.6), we get

$$\begin{aligned} \sum_{j=N+1}^{\infty} m(\lambda_j) e^{-\text{Re}(\lambda_j)/2} &\leq \sum_{k=N+1}^{\infty} \sum_{k \leq \text{Re}(\lambda_j) \leq k+1} m(\lambda_j) e^{-\text{Re}(\lambda_j)/2} \\ &\leq \sum_{k=N+1}^{\infty} \mathcal{N}(k+1) e^{-k/2} \\ &\leq \sum_{k=N+1}^{\infty} C_1 (k+1)^{d/2} e^{-k/2} < \infty, \end{aligned} \quad (8.7)$$

where C_1 is a positive constant.

Hence, by (8.4), (8.5), (8.7) and the definition of the theta function, we have that given a positive number $C > 0$, there exist a positive integer N and $K > 0$ such that

$$\left| \theta(t) - \sum_{j=1}^N m(\lambda_j) e^{-t\lambda_j} \right| \leq K e^{-Ct}, \quad t \geq 1. \quad (8.8)$$

Furthermore, by the asymptotic expansion of the trace of the operator $e^{-tA_\chi^\sharp(\sigma)}$ (8.1), we have that for every positive integer N ,

$$\theta(t) - \sum_{j=0}^N a_j t^{j-\frac{d}{2}} = O(t^{N-\frac{d}{2}}), \quad t \rightarrow 0.$$

All in all, we have proved that $\theta(t)$ satisfies the assumptions as in [JL93, AS 1, AS 2, p. 16]. Hence, we can apply [JL93, Theorem 1.5] for $p = j - \frac{d}{2}$ and obtain the meromorphic continuation of the xi function. The simple poles are located at $k_j = \frac{d}{2} - j$ with $\text{res}(k_j, \xi(\cdot, s; \sigma)) = a_j$. \square

Let $N(0) \subset \mathbb{C}$ be a neighborhood of zero in \mathbb{C} .

Theorem 8.4. *For every $s \in N(0)$, the xi function $\xi(z, s; \sigma)$ is holomorphic at $z = 0$.*

Proof. See [JL93, Theorem 1.6]. \square

The generalized zeta function is by definition the xi function divided by $\Gamma(z)$:

$$\zeta(z, s; \sigma) = \frac{1}{\Gamma(z)} \xi(z, s; \sigma). \quad (8.9)$$

Consequently, it is also holomorphic at $z = 0$. It holds

$$\left. \frac{d}{dz} \zeta(z, s; \sigma) \right|_{z=0} = \xi(0, s; \sigma). \quad (8.10)$$

Definition 8.5. The regularized determinant of the operator $A_\chi^\sharp(\sigma) + s^2$ is defined by

$$\det(A_\chi^\sharp(\sigma) + s^2) := \exp \left(- \left. \frac{d}{dz} \zeta(z, s; \sigma) \right|_{z=0} \right). \quad (8.11)$$

By (8.10) and (8.11) we get

$$\det(A_\chi^\sharp(\sigma) + s^2) = \exp(-\xi(0, s; \sigma)).$$

Equivalently,

$$\log(\det(A_\chi^\sharp(\sigma) + s^2)) = -\xi(0, s; \sigma). \quad (8.12)$$

Theorem 8.6. Let $\det(A_\chi^\sharp(\sigma) + s^2)$ be the regularized determinant associated to the operator $A_\chi^\sharp(\sigma) + s^2$. Then,

1. **case(a)** the Selberg zeta function has the representation

$$Z(s; \sigma, \chi) = \det(A_\chi^\sharp(\sigma) + s^2) \exp \left(-2\pi \dim(V_\chi) \text{Vol}(X) \int_0^s P_\sigma(t) dt \right). \quad (8.13)$$

2. **case(b)** the symmetrized zeta function has the representation

$$S(s; \sigma, \chi) = \det(A_\chi^\sharp(\sigma) + s^2) \exp \left(-4\pi \dim(V_\chi) \text{Vol}(X) \int_0^s P_\sigma(t) dt \right). \quad (8.14)$$

Proof. By the generalized resolvent identity (6.1) and estimates (6.8), we can proceed as in the proof of Proposition 6.5 to get

$$\text{Tr} \prod_{i=1}^N (A_\chi^\sharp(\sigma) + s_i^2)^{-1} = \int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \text{Tr}(e^{-tA_\chi^\sharp(\sigma)}) dt.$$

We have

$$\begin{aligned}
& \int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \operatorname{Tr}(e^{-tA_\chi^\sharp(\sigma)}) dt \\
&= \int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i t} \left(-\frac{d}{ds_i} e^{-ts_i^2} \right) \operatorname{Tr}(e^{-tA_\chi^\sharp(\sigma)}) dt.
\end{aligned} \tag{8.15}$$

For $\operatorname{Re}(z) > d/2$, we consider the limit as $z \rightarrow 0$

$$\begin{aligned}
& \lim_{z \rightarrow 0} \int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \left(-\frac{d}{ds_i} e^{-ts_i^2} \right) t^{z-1} \operatorname{Tr}(e^{-tA_\chi^\sharp(\sigma)}) dt \\
&= \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \frac{d}{ds_i} \int_0^\infty -e^{-ts_i^2} t^{-1} \operatorname{Tr}(e^{-tA_\chi^\sharp(\sigma)}) dt.
\end{aligned}$$

Hence, the right hand side of (8.15) gives

$$\begin{aligned}
& \int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i t} \left(-\frac{d}{ds_i} e^{-ts_i^2} \right) \operatorname{Tr}(e^{-tA_\chi^\sharp(\sigma)}) dt \\
&= \lim_{z \rightarrow 0} \int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \left(-\frac{d}{ds_i} e^{-ts_i^2} \right) t^{z-1} \operatorname{Tr}(e^{-tA_\chi^\sharp(\sigma)}) dt \\
&= \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \frac{d}{ds_i} \int_0^\infty -e^{-ts_i^2} t^{-1} \operatorname{Tr}(e^{-tA_\chi^\sharp(\sigma)}) dt \\
&= \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \frac{d}{ds_i} \left(-\xi(0, s_i; \sigma) \right) \\
&= \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \frac{d}{ds_i} \left(\log \left(\det(A_\chi^\sharp(\sigma) + s^2) \right) \right),
\end{aligned}$$

where in the last equation we used (8.12). Therefore, (8.15) becomes

$$\begin{aligned} & \int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \operatorname{Tr}(e^{-tA_\chi^\sharp(\sigma)}) dt \\ &= \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \frac{d}{ds_i} \log \det(A_\chi^\sharp(\sigma) + s_i^2) \end{aligned} \quad (8.16)$$

We treat here the case (b). One can proceed similarly for the case (a). The left-hand side of (8.16) can be developed more, if we insert the trace formula (4.46) for the operator $e^{-tA_\chi^\sharp(\sigma)}$. We have

$$\begin{aligned} \int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \operatorname{Tr}(e^{-tA_\chi^\sharp(\sigma)}) dt &= \int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \\ & e^{-ts_i^2} \left(2 \dim(V_\chi) \operatorname{Vol}(X) \int_{\mathbb{R}} e^{-t\lambda^2} P_\sigma(i\lambda) d\lambda \right. \\ & \left. + \sum_{[\gamma] \neq e} \frac{l(\gamma) \operatorname{tr}(\chi(\gamma))}{n_\Gamma(\gamma)} L_{\text{sym}}(\gamma; \sigma + w\sigma) \frac{e^{-l(\gamma)^2/4t}}{(4\pi t)^{1/2}} \right) dt. \end{aligned}$$

We use Lemma 6.4 to interchange the order of integration for the double integral

$$\int_0^\infty \int_{\mathbb{R}} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} e^{-t\lambda^2} P_\sigma(i\lambda) d\lambda.$$

As in Section 6.3, we can use the Cauchy integral formula to calculate this integral. Then, we obtain equation (6.19). For the calculation of the integral that corresponds to the hyperbolic contribution, we make again use of the identity (cf. [EMOT54, p. 146, (27)])

$$\int_0^\infty e^{-ts^2} \frac{e^{-l(\gamma)^2/4t}}{(4\pi t)^{1/2}} dt = \frac{1}{2s} e^{-sl(\gamma)}.$$

Hence,

$$\begin{aligned}
& \int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \left(2 \dim(V_\chi) \text{Vol}(X) \int_{\mathbb{R}} e^{-t\lambda^2} P_\sigma(i\lambda) d\lambda \right. \\
& \quad \left. + \sum_{[\gamma] \neq e} \frac{l(\gamma) \text{tr}(\chi(\gamma))}{n_\Gamma(\gamma)} L_{sym}(\gamma; \sigma + w\sigma) \frac{e^{-l(\gamma)^2/4t}}{(4\pi t)^{1/2}} \right) dt \\
& = \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{2\pi}{s_i} \dim(V_\chi) \text{Vol}(X) P_\sigma(s_i) \\
& \quad + \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \sum_{[\gamma] \neq e} \frac{l(\gamma) \text{tr}(\chi(\gamma))}{n_\Gamma(\gamma)} L_{sym}(\gamma; \sigma + w\sigma) e^{-s_i l(\gamma)}.
\end{aligned}$$

By (8.16), we get

$$\begin{aligned}
& \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \frac{d}{ds_i} \log \det(A_\chi^\#(\sigma) + s_i^2) \\
& = \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{2\pi}{s_i} \dim(V_\chi) \text{Vol}(X) P_\sigma(s_i) \\
& \quad + \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \sum_{[\gamma] \neq e} \frac{l(\gamma) \text{tr}(\chi(\gamma))}{n_\Gamma(\gamma)} L_{sym}(\gamma; \sigma + w\sigma) e^{-s_i l(\gamma)}.
\end{aligned} \tag{8.17}$$

We fix now the variables $s_2, \dots, s_N \in \mathbb{C}$ and let the variable $s_1 = s \in \mathbb{C}$ vary. Then, we can remove the structure

$$\sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right),$$

and get

$$\begin{aligned}
\frac{d}{ds} \log \det(A_\chi^\#(\sigma) + s^2) & = 4\pi \dim(V_\chi) \text{Vol}(X) P_\sigma(s) \\
& \quad + \sum_{[\gamma] \neq e} \frac{l(\gamma) \text{tr}(\chi(\gamma))}{n_\Gamma(\gamma)} L_{sym}(\gamma; \sigma) e^{-sl(\gamma)} \\
& \quad + K'(s),
\end{aligned} \tag{8.18}$$

where $K'(s)$ is a certain odd polynomial, which is of the form

$$K'(s) = \prod_{j=2}^N (s_j^2 - s^2) 2s Q(s_2, \dots, s_N).$$

The quantity $Q(s_2, \dots, s_N)$ comes from the terms that correspond to the summands over $i = 2, \dots, N$ and hence it has a fixed value in \mathbb{C} , since s_2, \dots, s_N are fixed.

Next, we can substitute the term that comes from the hyperbolic distribution of the trace formula with the logarithmic derivative of the symmetrized zeta function. By (2.25) we have

$$\begin{aligned} \frac{d}{ds} \log \det(A_\chi^\sharp(\sigma) + s^2) &= 4\pi \dim(V_\chi) \text{Vol}(X) P_\sigma(s) \\ &\quad + L_S(s) + K'(s). \end{aligned}$$

We integrate with respect to s and we get

$$\begin{aligned} \log \det(A_\chi^\sharp(\sigma) + s^2) &= 4\pi \dim(V_\chi) \text{Vol}(X) \int_0^s P_\sigma(t) dt \\ &\quad + \log S(s; \sigma, \chi) + K(s). \end{aligned}$$

Hence,

$$\begin{aligned} \log S(s; \sigma, \chi) &= \log \det(A_\chi^\sharp(\sigma) + s^2) - K(s) \\ &\quad - 4\pi \dim(V_\chi) \text{Vol}(X) \int_0^s P_\sigma(t) dt. \end{aligned} \quad (8.19)$$

We want to show that $K(s) = 0$. For that reason, we study the asymptotic behavior of all terms in equation (8.19), as $s \rightarrow \infty$. By (2.25), $\log S(z, s)$ decreases exponentially as $s \rightarrow \infty$.

We use now the asymptotic expansion of $\log \det(A_\chi^\sharp(\sigma) + s_i^2)$ as $s \rightarrow \infty$, as it is described in [QHS93, p. 219-220]. We write the short time asymptotic expansion (8.1) of the trace of the operator $e^{-tA^\sharp \tau, \chi(\sigma)}$ as

$$\text{Tr}(e^{-tA^\sharp \chi(\sigma)}) \sim_{t \rightarrow 0} \sum_{\nu=0}^{\infty} c_{j_\nu} t^{j_\nu},$$

where $j_\nu = j - \frac{d}{2}$, and we use formula [QHS93, equation (13)]. In our case, there are no coefficients $c_{j'_\nu}$ that corresponds to integers j'_ν , because d is odd and hence $j'_\nu = 0$. We have

$$\log \det(A_\chi^\sharp(\sigma) + s^2) \sim_{s \rightarrow \infty} \sum_{k=0}^{\infty} c_{(2k-d)/2} \Gamma((2k-d)/2) s^{d-2k}. \quad (8.20)$$

The right hand side of (8.20) contains only odd powers of s . On the other hand, the Plancherel polynomial is an even polynomial of s . Regarding (8.19), as $s \rightarrow \infty$, we have that an odd polynomial equals an even one. Therefore, the coefficients $c_{(2k-d)/2}$ vanish, as well as the coefficients of the even polynomial $K(s)$.

Finally, exponentiating equation (8.19) for $K(s) = 0$ we obtain

$$S(s; \sigma, \chi) = \det(A_\chi^\sharp(\sigma) + s^2) \exp \left(-4\pi \dim(V_\chi) \text{Vol}(X) \int_0^s P_\sigma(t) dt \right). \quad (8.21)$$

□

We prove now a determinant formula for the Ruelle zeta function. We define the operator

$$A_\chi^\sharp(\sigma_p \otimes \sigma) := \bigoplus_{\sigma' \in \widehat{M}} \bigoplus_{i=1}^{[(\sigma_p \otimes \sigma):\sigma']} A_\chi(\sigma') \quad (8.22)$$

acting on the space $C^\infty(X, E(\sigma') \otimes E_\chi)$, where $\sigma \in \widehat{M}$, $E(\sigma')$ is the vector bundle over X , constructed as in Section 4.3 and σ_p denotes the p -th exterior power of the standard representation of M . We distinguish again two cases for $\sigma' \in \widehat{M}$.

- **case (a)**: σ' is invariant under the action of the restricted Weyl group W_A . Then, $i^*(\tau) = \sigma'$, where $\tau \in R(K)$.
- **case (b)**: σ' is not invariant under the action of the restricted Weyl group W_A . Then, $i^*(\tau) = \sigma' + w\sigma'$, where $\tau \in R(K)$.

Proposition 8.7. *The Ruelle zeta function has the representation*

- **case (a)**

$$R(s; \sigma, \chi) = \prod_{p=0}^{d-1} \det(A_\chi^\sharp(\sigma_p \otimes \sigma) + (s + \rho - \lambda)^2)^{(-1)^p} \exp \left(-2\pi(d+1) \dim(V_\chi) \dim(V_\sigma) \text{Vol}(X)s \right). \quad (8.23)$$

- **case (b)**

$$R(s; \sigma, \chi)R(s; w\sigma, \chi) = \prod_{p=0}^{d-1} \det(A_\chi^\sharp(\sigma_p \otimes \sigma) + (s + \rho - \lambda)^2)^{(-1)^p} \exp \left(-4\pi(d+1) \dim(V_\chi) \dim(V_\sigma) \text{Vol}(X)s \right). \quad (8.24)$$

Proof. We prove the assertion for case (b). One can proceed similarly for case (a). By Proposition 6.10, we have the expression of the Ruelle zeta function as a product of Selberg zeta functions. Then, we see

$$\begin{aligned} R(s; \sigma, \chi)R(s; w\sigma, \chi) &= \prod_{p=0}^{d-1} Z(s + \rho - \lambda; \sigma_p \otimes \sigma, \chi)^{(-1)^p} \prod_{p=0}^{d-1} Z(s + \rho - \lambda; \sigma_p \otimes w\sigma, \chi)^{(-1)^p} \\ &= \prod_{p=0}^{d-1} S(s + \rho - \lambda; \sigma_p \otimes \sigma, \chi)^{(-1)^p}. \end{aligned}$$

Hence, if we equip the determinant formula for the symmetrized zeta function (Theorem 8.6.2), we have

$$\begin{aligned} R(s; \sigma, \chi)R(s; w\sigma, \chi) &= \prod_{p=0}^{d-1} \det(A_\chi^\sharp(\sigma_p \otimes \sigma) + (s + \rho - \lambda)^2)^{(-1)^p} \\ &\quad \exp\left(\sum_{p=0}^{d-1} (-1)^p (-4\pi \dim(V_\chi) \text{Vol}(X)) \int_0^{s+\rho-\lambda} P_{\sigma_p \otimes \sigma}(t) dt\right). \end{aligned} \tag{8.25}$$

On the other hand,

$$\sum_{p=0}^{d-1} (-1)^p \int_0^{s+\rho-\lambda} P_{\sigma_p \otimes \sigma}(t) dt = \int_0^s f(t) dt,$$

where $f(t)$ is defined as in (7.19). Therefore, by Lemma 7.8,

$$\sum_{p=0}^{d-1} (-1)^p \int_0^{s+\rho-\lambda} P_{\sigma_p \otimes \sigma}(t) dt = (d+1) \dim(V_\sigma) s. \tag{8.26}$$

We substitute equation (8.26) in (8.25) and we get

$$\begin{aligned} R(s; \sigma, \chi)R(s; w\sigma, \chi) &= \prod_{p=0}^{d-1} \det(A_\chi^\sharp(\sigma_p \otimes \sigma) + (s + \rho - \lambda)^2)^{(-1)^p} \\ &\quad \exp\left(-4\pi(d+1) \dim(V_\chi) \dim(V_\sigma) \text{Vol}(X) s\right). \end{aligned}$$

□

CHAPTER 9

Discussion

9.1 Ray-Singer analytic torsion

In this section we will recall the definition of the analytic torsion $T_X^{RS}(\chi; E_\chi)$. Let X be an oriented compact odd dimensional riemannian manifold. We let (χ, V_χ) be a finite dimensional representation of Γ . Let E_χ be the associated flat vector bundle over X . We choose a hermitian metric h in E_χ . Let g be the riemannian metric on X .

In general, the analytic torsion does depend on the riemannian metric g and the hermitian metric h . However, by [Mül93, Corollary 2.7], if $\dim X$ is an odd integer and χ is considered to be acyclic (cf. Assumption 9.3), then, the analytic torsion does not depend on g and h . Hence, instead of $T_X^{RS}(\chi; E_\chi)(g, h)$ we simply write $T_X^{RS}(\chi; E_\chi)$.

We follow [RS71, p.148-151] and [MM63, p.370-372]. Let $\Lambda^p(X, E_\chi)$ be the space of smooth differential p -forms on X with values in E_χ . We define the space $C^\infty(\Gamma \backslash G, \Lambda^p \mathfrak{p}^* \otimes V_\chi)$ by

$$C^\infty(\Gamma \backslash G, \Lambda^p \mathfrak{p}^* \otimes V_\chi) := \{f \in C^\infty(G) : f(kg) = \mu_p^{-1}(k)g, \forall g \in G, \forall k \in K, \\ f(\gamma g) = f(g), \forall g \in G, \forall \gamma \in \Gamma\},$$

where μ_p denotes the p -th exterior power of the Ad^* -representation of K in \mathfrak{p} , i.e.

$$\mu_p := \Lambda^p \text{Ad}^* : K \rightarrow \text{GL}(\Lambda^p \mathfrak{p}^*).$$

Then, there exists an isomorphism

$$\Lambda^p(X, E_\chi) \cong C^\infty(\Gamma \backslash G, \Lambda^p \mathfrak{p}^* \otimes V_\chi)$$

(see [MP13, p. 12, and p. 15])

Let ϕ be a p -differential form on the universal covering \tilde{X} with values in E_χ . Then, $\gamma \in \Gamma$ acts on ϕ by $\gamma^*\phi = \chi(\gamma)\phi$, where the element γ acts on \tilde{X} by deck transformations. Let $d_\chi : \Lambda^p(X, E_\chi) \rightarrow \Lambda^{p+1}(X, E_\chi)$ be the exterior derivative operator. Since E_χ is flat, $d_\chi \circ d_\chi = 0$. Hence, we obtain a de Rham complex

$$\Lambda^0(X, E_\chi) \xrightarrow{d_\chi} \Lambda^1(X, E_\chi) \xrightarrow{d_\chi} \dots \xrightarrow{d_\chi} \Lambda^d(X, E_\chi).$$

We denote by $H^p(X; E_\chi)$ the p -th cohomology group of this complex.

We want to describe $\phi \in \Lambda^p(X, E_\chi)$ locally. Let $(E_\chi)_x$ be the fiber over $x \in X$ and $(E_\chi)_x^*$ its dual vector space. We denote by $\langle \cdot, \cdot \rangle$ the canonical bilinear pairing on $E_\chi \times E_\chi^*$. For $x \in X$, let $(dx^1, dx^2, \dots, dx^d)$ be the canonical basis of T_x^*X , consisting of \mathbb{C} -valued 1-differential forms on X , associated with a local coordinate system (x^1, x^2, \dots, x^d) . Then, ϕ can be written as

$$\phi = \sum_{i_1 < \dots < i_p} u_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p},$$

where $u_{i_1 \dots i_p}$ are smooth sections of $(E_\chi)_x$. Let ω be a q -differential form with values in the dual vector bundle E_χ^* . This means that ω can be written as

$$\omega = \sum_{j_1 < \dots < j_q} v_{j_1 \dots j_q} dx^{j_1} \wedge \dots \wedge dx^{j_q},$$

where $v_{j_1 \dots j_q}$ are smooth sections of $(E_\chi)_x^*$. We define the wedge product by the $(p+q)$ -differential form, given by

$$\phi \wedge \bar{\omega} = \sum_{i_1 < \dots < i_p} \sum_{j_1 < \dots < j_q} \langle u_{i_1 \dots i_p}, v_{j_1 \dots j_q} \rangle dx^{i_1} \wedge \dots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_q}.$$

We define the isomorphism $\sharp : E_\chi \rightarrow E_\chi^*$, induced by the metric h , as

$$\sharp(v_x)(u_x) := \langle \sharp(v_x), u_x \rangle = h(v_x, u_x),$$

where $v_x \in E_\chi$, $u_x \in (E_\chi)_x^*$. We extend this isomorphism to

$$\sharp : \Lambda^p(X, E_\chi) \rightarrow \Lambda^p(X, E_\chi^*).$$

The riemannian metric on X defines an operator

$$* : \Lambda^p(X, E_\chi) \rightarrow \Lambda^{d-p}(X, E_\chi),$$

acting as

$$*\phi = \sum_{i_1 < \dots < i_p} u_{i_1 \dots i_p} * (dx^{i_1} \wedge \dots \wedge dx^{i_p}),$$

where $*$ in the right hand side of the equation above acts as the usual $*$ -operator on \mathbb{C} -valued differential forms on X . For every $p = 0, \dots, d$, we have $** = (-1)^{p(d-p)} \text{Id}$ on $\Lambda^p(X, E_\chi)$. We consider the following composition

$$* \circ \sharp := (* \otimes \text{Id}) \circ (\text{Id} \otimes \sharp) : \Lambda^p T^* X \otimes E_\chi \rightarrow \Lambda^{d-p} T^* X \otimes E_\chi^*.$$

We define the inner product on $\Lambda^p(X, E_\chi)$ by

$$(\theta, \phi) := \int_X \theta \wedge \overline{* \circ \sharp \phi}. \quad (9.1)$$

Then, the formal adjoint of d_χ with respect to the inner product (9.1) is the operator δ on $\Lambda^p(X, E_\chi)$ given by

$$\delta_\chi = (-1)^{d(p+1)+1} * \circ \sharp^{-1} \circ d_\chi \circ \sharp \circ *.$$

We define the Hodge-Laplace operator $\Delta_{\chi,p} : \Lambda^p(X, E_\chi) \hookrightarrow$ by

$$\Delta_{\chi,p} := d_\chi \delta_\chi + \delta_\chi d_\chi. \quad (9.2)$$

The operator $\Delta_{\chi,p}$ is an elliptic positive essentially self-adjoint operator. Let $\mathcal{H}^p(X, E_\chi) := \ker(\Delta_{\chi,p})$ be the subspaces of the p -harmonic forms. By Hodge theory, we have the isomorphism

$$H^p(X; E_\chi) \cong \mathcal{H}^p(X, E_\chi).$$

Let $\Delta'_{\chi,p}$ be the restriction of the Hodge-Laplacian $\Delta_{\chi,p}$ to the orthogonal complement of $\mathcal{H}^p(X, E_\chi)$ with respect to the inner product (9.1). We define the zeta function $\zeta_{\Delta'_{\chi,p}}(z)$ of $\Delta'_{\chi,p}$ by

$$\zeta_{\Delta'_{\chi,p}}(z) := \text{Tr}(\Delta'_{\chi,p})^{-z},$$

for $\text{Re}(z) > d/2$ (cf. Appendix A, Definition A.6). It is a well-known fact (cf. [Gil95, Lemma 1.10.1]) that $\zeta_{\Delta'_{\chi,p}}(z)$ admits a meromorphic continuation to the whole complex plane \mathbb{C} and is regular at $s = 0$. Similarly to the Definition 8.5, we define the regularized determinant of $\Delta'_{\chi,p}$ by

$$\det(\Delta'_{\chi,p}) := \exp \left(- \frac{d}{dz} \zeta_{\Delta'_{\chi,p}}(z) \Big|_{z=0} \right). \quad (9.3)$$

Definition 9.1. We define the Ray-Singer analytic torsion associated with a finite dimensional unitary complex representation of Γ by the formula

$$\log T_X^{RS}(\chi; E_\chi) := \frac{1}{2} \sum_{p=0}^d (-1)^p p \zeta'_{\Delta'_{\chi,p}}(0). \quad (9.4)$$

Equivalently, by (9.3) the analytic torsion $T_X^{RS}(\chi; E_\chi)$ can be expressed by the regularized determinants of the Laplacians $\Delta'_{\chi,p}$:

$$T_X^{RS}(\chi; E_\chi) = \prod_{p=0}^d (\det(\Delta'_{\chi,p}))^{-\frac{1}{2}p(-1)^p}. \quad (9.5)$$

We mention here that since $\Delta'_{\chi,p}$ is a positive essentially self-adjoint operator, the analytic torsion is a positive real number, i.e. $T_X^{RS}(\chi; E_\chi) \in \mathbb{R}^+$.

9.2 Refined analytic torsion

Let $\chi : \Gamma \rightarrow \text{GL}(V_\chi)$ be a finite dimensional complex representation of Γ . Let E_χ be the associated flat vector bundle over X . Let ∇ be the flat connection on E_χ . As in the previous section, we denote by $\Lambda^p(X, E_\chi)$ the space of all smooth p -differential forms on X with values in E_χ .

The refined analytic torsion $T_X^{\mathbb{C}}(\chi; E_\chi)$ associated with the representation χ of Γ , as its name declares, is a refinement of the Ray-Singer analytic torsion $T_X^{RS}(\chi; E_\chi)$. Whereas the Ray-Singer analytic torsion is a positive real number, the refined analytic torsion is, in general, a complex number.

If χ is unitary, then the refined analytic torsion can be expressed as the product of the Ray-Singer analytic torsion and a phase factor, which involves the eta invariant of the odd signature operator (cf. Definition 9.2). Hence, in this case the absolute value of the refined analytic torsion is equal to the Ray-Singer analytic torsion (cf. Remark 9.8 below).

We begin with the following definitions as they are given in [BK05] and [BK08]. Let $d = 2n + 1, n \in \mathbb{N}$ be the dimension of X . Let $*$: $\Lambda^k(X, E_\chi) \rightarrow \Lambda^{d-k}(X, E_\chi)$ be the $*$ -operator with respect to the riemannian metric g . Let Γ be the operator defined by $\Gamma := i^n(-1)^{k(k+1)/2}*$, acting on $\Lambda^k(X, E_\chi)$.

Definition 9.2. We define the odd signature operator $B = B(\nabla, g) : \Lambda^k(X, E_\chi) \rightarrow \Lambda^k(X, E_\chi)$ by

$$B = \Gamma\nabla + \nabla\Gamma.$$

Explicitly, for a $\omega \in \Lambda^p(X, E_\chi)$ one has

$$B\omega = i^n(-1)^{p(p+1)/2}(*\nabla - \nabla*)\omega \in \Lambda^{d-p-1}(X, E_\chi) \oplus \Lambda^{d-p+1}(X, E_\chi). \quad (9.6)$$

The odd signature operator is an elliptic operator of order 1, but no longer self-adjoint. This is because we assumed that χ is non-unitary and hence the corresponding flat connection ∇ is not hermitian, i.e. there is no hermitian metric h in E_χ , which is compatible with ∇ . Nevertheless, it has a self-adjoint principal

symbol, and hence it has nice spectral properties. Namely, its spectrum is discrete and contained in a translate of a cone in \mathbb{C} (cf. Appendix A). In addition, we can define as in Definition 5.4, the eta function and the eta invariant of the operator B . We set

$$\begin{aligned}\Lambda^{even}(X, E_\chi) &:= \bigoplus_{p=0}^{n-1} \Lambda^{2p}(X, E_\chi) \\ B_{even} &:= \bigoplus_{p=0}^{n-1} B_{2p} : \Lambda^{even}(X, E_\chi) \rightarrow \Lambda^{even}(X, E_\chi),\end{aligned}$$

where B_{2p} denotes the operator B acting on $2p$ -differential forms. We call the operator B_{even} the *even part of B* . Since it acts in differential forms of even degree, it can be slightly simplified. We have for a $\omega \in \Lambda^{2p}(X, E_\chi)$,

$$B^{even}\omega = i^n(-1)^{(p+1)}(*\nabla - \nabla*)\omega \in \Lambda^{d-2p-1}(X, E_\chi) \oplus \Lambda^{d-2p+1}(X, E_\chi). \quad (9.7)$$

In order to define the refined analytic torsion we need the following two assumptions.

Assumption 9.3. *The representation χ of Γ is acyclic, i.e., $H^p(X; E_\chi) = 0$, for every $p = 0, \dots, d$.*

Assumption 9.4. *The even part B^{even} of the odd signature operator is bijective.*

We define $\Lambda_+^k(X, E_\chi) := \ker(\Delta\Gamma) \cap \Lambda^k(X, E_\chi)$ and $\Lambda_-^k(X, E_\chi) := \ker(\Delta\Gamma) \cap \Lambda^k(X, E_\chi)$, where $k = 0, \dots, d$. By Assumption 9.4, we have that $\Lambda^k(X, E_\chi) = \Lambda_+^k(X, E_\chi) \oplus \Lambda_-^k(X, E_\chi)$, and hence we obtain a grading on $\Lambda^{2p}(X, E_\chi)$. We put

$$\begin{aligned}\Lambda_+^{2p}(X, E_\chi) &:= \ker(\Delta\Gamma) \cap \Lambda^{2p}(X, E_\chi) \\ \Lambda_-^{2p}(X, E_\chi) &:= \ker(\Gamma\Delta) \cap \Lambda^{2p}(X, E_\chi).\end{aligned}$$

We define

$$\Lambda_\pm^{even}(X, E_\chi) := \bigoplus_{p=0}^d \Lambda_\pm^{2p}(X, E_\chi),$$

and let B_\pm^{even} be the restriction of B^{even} to $\Lambda_\pm^{even}(X, E_\chi)$. Then, B^{even} leaves the subspaces $\Lambda_\pm^{even}(X, E_\chi)$ invariant. It follows from Assumption 9.4 that the operators

$$B_\pm^{even} : \Lambda_\pm^{even}(X, E_\chi) \rightarrow \Lambda_\mp^{even}(X, E_\chi)$$

are bijective.

Let $\theta \in (-\pi, 0)$ be an Agmon angle for B_{even} . Then, θ is an Agmon angle for B_{\pm}^{even} as well. The graded determinant of the operator B^{even} is a non-zero complex number defined by

$$\det_{gr,\theta}(B^{even}) := \frac{\det_{\theta}(B_{+}^{even})}{\det_{\theta}(B_{-}^{even})}.$$

We set

$$\xi_{\chi} = \xi_{\chi}(\nabla, g, \theta) := \frac{1}{2} \sum_{k=0}^d (-1)^k \zeta'_{2\theta}(0, (\Gamma\nabla)^2|_{\Lambda_{+}^k(X, E_{\chi})}), \quad (9.8)$$

where $\zeta'_{2\theta}(0, (\Gamma\nabla)^2|_{\Lambda_{+}^k(X, E_{\chi})})$ is the derivative with respect to z of the zeta function, of the operator $(\Gamma\nabla)^2|_{\Lambda_{+}^k(X, E_{\chi})}$ corresponding to the spectral cut along the ray R_{θ} (cf. Appendix A, Definition A.2).

Theorem 9.5. *Let $\theta \in (-\pi/2, 0)$ be an Agmon angle for B_{even} such that there are no eigenvalues of B_{even} in the solid angles $L_{(-\pi/2, \theta]}$ and $L_{(\pi/2, \theta+\pi]}$. Then*

$$\det_{gr,\theta}(B^{even}) = e^{\xi_{\chi}} e^{-i\pi\eta(B^{even})}.$$

Proof. This is proved in [BK08, Theorem 7.2]. □

Definition 9.6. Let $\chi : \Gamma \rightarrow \text{GL}(V_{\chi})$ be a finite dimensional complex representation of Γ , such that Assumptions 9.3 and 9.4 are satisfied. Let the operator $B^{even} : \omega \in \Lambda^{2p}(X, E_{\chi}) \rightarrow \Lambda^{d-2p-1}(X, E_{\chi}) \oplus \Lambda^{d-2p+1}(X, E_{\chi})$ be as in (9.7), and $\theta \in (-\pi, 0)$ be an Agmon angle for B^{even} . Then, we define the refined analytic torsion $T_{\chi}^{\mathbb{C}}(X; E_{\chi}) \in \mathbb{C} - \{0\}$ by the formula

$$T_{\chi}^{\mathbb{C}}(X; E_{\chi}) := \det_{gr,\theta}(B^{even}) e^{i\pi \text{rank}(E_{\chi}) \eta_{triv}(B_{triv}^{even})}, \quad (9.9)$$

where $\eta_{trivial}(B_{triv})$ denotes the eta invariant of the even part of the odd signature operator B_{triv} , associated with the trivial connection on the trivial line bundle $E_{\chi, triv}$ over X .

By [APS75], if $\dim X \equiv 1 \pmod{4}$, then $\eta_{triv}(B_{triv}) = 0$. Then, we obtain

$$T_{\chi}^{\mathbb{C}}(X; E_{\chi}) := \det_{gr,\theta}(B^{even}). \quad (9.10)$$

In general, the refined analytic torsion does depend on the riemannian metric g , as well as on ∇ and θ . Nevertheless, if we consider the set $\mathcal{M}(\nabla)$ of riemannian metrics g that they are admissible for ∇ , i.e. the operator B^{even} satisfies Assumption 9.4, then for an acyclic representation χ of Γ (that is Assumption 9.3 is satisfied), $\mathcal{M}(\nabla)$ is non-empty.

Theorem 9.7. *Let $E_\chi \rightarrow X$ be a flat vector bundle associated with an acyclic finite dimensional representation χ of Γ over a closed oriented odd dimensional manifold X . Let ∇ be the flat connection on E_χ . For each $g \in \mathcal{M}(\nabla)$, the refined analytic torsion $T_\chi^{\mathbb{C}}(X; E_\chi)$ is independent of the riemannian metric g and the Agmon angle θ .*

Proof. See [BK08, Theorem 9.3]. □

Hence, we write $T_\chi^{\mathbb{C}}(X; E_\chi) \equiv T_\chi^{\mathbb{C}}(X; E_\chi)(g, \nabla, \theta)$.

Remark 9.8. *If the representation χ of Γ is unitary then the expression of ξ_χ in (9.8) coincides with the expression of the logarithm of the analytic torsion $T_\chi^{RS}(X; E_\chi)$ in (9.4), i.e.*

$$\xi_\chi = \log T_\chi^{RS}(X; E_\chi),$$

Hence, by Theorem 9.5,

- if $\dim X \equiv 1 \pmod{4}$,

$$T_\chi^{\mathbb{C}}(X; E_\chi) = T_\chi^{RS}(X; E_\chi) e^{-i\pi\eta(B^{even})};$$

- if $\dim X \equiv 3 \pmod{4}$,

$$T_\chi^{\mathbb{C}}(X; E_\chi) = T_\chi^{RS}(X; E_\chi) e^{-i\pi\eta(B^{even})} e^{i\pi \text{rank}(E_\chi) \eta_{triv}(B_{triv}^{even})}.$$

If the representation χ of Γ is not unitary, then by [BK08, Theorem 8.2] we have

$$\text{Re}(\xi_\chi) = \log T_\chi^{RS}(X; E_\chi).$$

By Theorem 9.5, we get

$$|\det_{gr, \theta}(B^{even})| = T_\chi^{RS}(X; E_\chi) e^{\text{Im}(\pi\eta(B^{even}))}.$$

Since B_{triv} is self-adjoint, its eta invariant $\eta(B_{triv})$ is a real number (cf. [BK08, Section 9.1]). Hence, for every odd integer $d = \dim X$, we have

$$|T_\chi^{\mathbb{C}}(X; E_\chi)| = T_\chi^{RS}(X; E_\chi) e^{\text{Im}(\pi\eta(B^{even}))}.$$

Proposition 8.7 gives an interpretation of the Ruelle zeta function in terms of the determinants of the operators $A_{\tau, \chi}^\sharp(\sigma_p \otimes \sigma) + (s + \rho - \lambda)^2$, $p = 0, \dots, d$. By definition, the operators $A_{\tau, \chi}^\sharp(\sigma_p \otimes \sigma)$ act on the smooth sections of the vector bundle $E(\sigma') \otimes E_\chi$, where $E(\sigma')$ is constructed to have a grading

$$E(\sigma') = E^+(\sigma') \oplus E^-(\sigma')$$

as in Section 4.3. In an obvious way, the operators $A_{\tau,\chi}^\sharp(\sigma_p \otimes \sigma)$ preserve the grading. Therefore, the determinants that appear in (8.23) and (8.24) can be considered as graded determinants .

We consider an acyclic representation χ of Γ for defining the refined analytic torsion. If one could apply the well-known Hodge theory, then

$$H^p(X; E_\chi) \cong \mathcal{H}^p(X, E_\chi),$$

where $\mathcal{H}^p(X, E_\chi) := \ker(A_{\tau,\chi}^\sharp(\sigma_p \otimes \sigma) + (\rho - \lambda)^2)$. In this case, we could easily get that $\ker(A_{\tau,\chi}^\sharp(\sigma_p \otimes \sigma) + (\rho - \lambda)^2) = 0$. The situation now is that one can not conclude the triviality of the kernels of the operators. Hence, the regularity of the Ruelle zeta function at zero is not trivial.

Conjectural Equality

- case (a)

$$R(0; \sigma, \chi) = \prod_{p=0}^{d-1} \det(A_{\tau,\chi}^\sharp(\sigma_p \otimes \sigma) + (\rho - \lambda)^2)^{(-1)^p}$$

- case (b)

$$R(0; \sigma, \chi) = e^{\frac{i\pi}{2}\eta(D_{p,\chi}^\sharp(\sigma))} \prod_{p=0}^{d-1} \det(A_{\tau,\chi}^\sharp(\sigma_p \otimes \sigma) + (\rho - \lambda)^2)^{(-1)^p/2}. \quad (9.11)$$

By Proposition 8.7, we can easily see that once we have the regularity of the Ruelle zeta function at zero, then, we can take the limit as $s \rightarrow 0$ of the right hand side of (8.23) and (8.24). In case (b), one has to make use the fact that the Ruelle zeta function can be written as

$$R(s; \sigma, \chi) = \sqrt{R(s; \sigma, \chi)R(s; w\sigma, \chi)R^s(s; \sigma, \chi)},$$

and then use the fact that by the functional equation (7.41) for the super Ruelle zeta function, we have

$$R^s(0; \sigma, \chi) = e^{i\pi\eta(D_{p,\chi}^\sharp(\sigma))}.$$

We recall here that from Theorem 9.5 we have

$$\det_{gr,\theta}(B^{even}) = e^{\xi_X} e^{-i\pi\eta(B^{even})}. \quad (9.12)$$

If we compare equations (9.11) and (9.12), by the definition of the refined analytic torsion $T_\chi^C(X; E_\chi)$, we are motivated to consider the Ruelle zeta function at zero as a candidate for $T_\chi^C(X; E_\chi)$.

Spectral theory

In Chapter 5, we defined the twisted Dirac operator $D_\chi^\sharp(\sigma)$ acting on the space of the smooth sections of the product-vector bundle $E_{\tau_s(\sigma)} \otimes E_\chi$. As we have already mentioned, this operator is not self-adjoint. We recall equation (5.4) from Section 5.1, which describes the pullback of the twisted Dirac operator to \tilde{X} ,

$$\tilde{D}_\chi^\sharp(\sigma) = \tilde{D}(\sigma) \otimes \text{Id}_{V_\chi}. \quad (\text{A.1})$$

Therefore, we can write the principal symbol $\sigma_{D_\chi^\sharp(\sigma)}$ of $D_\chi^\sharp(\sigma)$

$$\sigma_{D_\chi^\sharp(\sigma)}(x, \xi) = (i\xi) \otimes \text{Id}_{(V_{\tau(\sigma)} \otimes V_\chi)_x}, \quad x \in X, \xi \in T_x^*X, \xi \neq 0.$$

Then, we see that $D_\chi^\sharp(\sigma)$ is an elliptic operator of first order and so we can study the second order elliptic differential operator $(D_\chi^\sharp(\sigma))^2$ and its spectral properties. Furthermore, we can define the operators $e^{-t(D_\chi^\sharp(\sigma))^2}$ and $D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2}$. Before dealing with the twisted Dirac operator $D_\chi^\sharp(\sigma)$, the corresponding semi-group $e^{-t(D_\chi^\sharp(\sigma))^2}$, and the operator $D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2}$, we give a more general setting.

Setting A.1. *Let $E \rightarrow X$ be a complex vector bundle over a smooth compact riemannian manifold X of dimension d . Let $D : C^\infty(X, E) \rightarrow C^\infty(X, E)$ be an elliptic differential operator of order $m \geq 1$. Let σ_D be its principal symbol.*

Definition A.2. A spectral cut is a ray

$$R_\theta := \{\rho e^{i\theta} : \rho \in [0, \infty]\},$$

where $\theta \in [0, 2\pi)$.

We have to choose a specific angle $\theta \in [0, 2\pi)$.

Definition A.3. The angle θ is a principal angle for an elliptic operator D if

$$\text{spec}(\sigma_D(x, \xi)) \cap R_\theta = \emptyset, \quad \forall x \in X, \forall \xi \in T_x^*X, \xi \neq 0.$$

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Definition A.4. We define the solid angle L_I associated with a closed interval I of \mathbb{R} by

$$L_I := \{\rho e^{i\theta} : \rho \in (0, \infty), \theta \in I\}.$$

Definition A.5. The angle θ is an Agmon angle for an elliptic operator D , if it is a principal angle for D and there exists $\varepsilon > 0$ such that

$$\text{spec}(D) \cap L_{[\theta-\varepsilon, \theta+\varepsilon]} = \emptyset.$$

We want to define the ζ -function and the ζ -regularized determinant of the operator D . Let θ be an Agmon angle for D . We assume in addition that D is invertible. Then, there exists a $r_0 > 0$ such that

$$\text{spec}(D) \cap \{z \in \mathbb{C} : |z| \leq 2r_0\} = \emptyset.$$

We consider the contour $\Gamma_{\theta, r_0} \subset \mathbb{C}$, defined as $\Gamma_{\theta, r_0} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where $\Gamma_1 = \{r e^{i\theta} : \infty > r \geq r_0\}$, $\Gamma_2 = \{r_0 e^{ia} : \theta \leq a \leq \theta - 2\pi\}$, $\Gamma_3 = \{r e^{i(\theta-2\pi)} : r_0 \leq r < \infty\}$. Let D_θ^{-z} be the operator defined by

$$D_\theta^{-z} = \frac{i}{2\pi} \int_{\Gamma_{\theta, r_0}} \lambda^{-z} (D - \lambda)^{-1} d\lambda,$$

where $\text{Re}(z) > 0$. It is a well defined bounded (pseudo)-differential operator (cf. [Shu87, Corollary 9.2, Chapter 10]). We fix an Agmon angle θ for D . Then, we write D^{-z} for D_θ^{-z} .

Definition A.6. We define the ζ -function $\zeta(z, D)$ of an elliptic differential operator of order $m \geq 1$ by

$$\zeta(z, D) := \text{Tr}(D^{-z}) = \int_X \text{tr} D^{-z}(x, x) dx, \quad \text{Re}(z) > \frac{d}{m}. \quad (\text{A.2})$$

The ζ -function has a meromorphic continuation to the whole complex plane \mathbb{C} and is regular at 0 ([Shu87, Theorem 13.1]).

Definition A.7. The ζ -regularized determinant of D is defined by the formula

$$\det_\theta(D) := \exp \left(- \left. \frac{d}{dz} \right|_{z=0} \zeta_\theta(z, D) \right). \quad (\text{A.3})$$

By formula (A.3), we can write

$$\log \det_{\theta}(D) \sim -\zeta'_{\theta}(0, D).$$

However, the logarithm $\log \det_{\theta}(D)$ is defined only up to a multiple of $2\pi i$, while $-\zeta'_{\theta}(0, D)$ is a well defined complex number. For that reason, whenever we write $\log \det_{\theta}(D)$, we will always refer to the particular value of the logarithm such that

$$\log \det_{\theta}(D) = -\zeta'_{\theta}(0, D).$$

We assume here in addition that D has a self-adjoint symbol $\sigma_D(x, \xi)$, $x \in X$, $\xi \in T_x^*X$, $\xi \neq 0$, with respect to a fiber metric on E . Hence, D has nice spectral properties.

Lemma A.8. *Let ε be an angle such that the principal symbol $\sigma_D(x, \xi)$ of D , for $\xi \in T_x^*X$, $\xi \neq 0$ does not take values in $L_{[-\varepsilon, \varepsilon]}$. Then, the spectrum $\text{spec}(D)$ of the operator D is discrete and for every $\varepsilon \in (0, \frac{\pi}{2})$ there exist $R > 0$ such that $\text{spec}(D)$ is contained in the set $B(0, R) \cup L_{[-\varepsilon, \varepsilon]} \subset \mathbb{C}$.*

Proof. The discreteness of the spectrum follows from [Shu87, Theorem 8.4]. For the second statement see [Shu87, Theorem 9.3]. \square

Let λ_k be an eigenvalue of D and V_{λ_k} be the corresponding eigenspace. This is a finite dimensional subspace of $C^{\infty}(X, E)$ invariant under D . We have that for every $k \in \mathbb{N}$, there exist $N_k \in \mathbb{N}$ such that

$$\begin{aligned} (D - \lambda_k \text{Id})^{N_k} V_{\lambda_k} &= 0 \\ \lim_{k \rightarrow \infty} |\lambda_k| &= \infty. \end{aligned}$$

By [Mar88] the space $L^2(X, E)$ can be decomposed as

$$L^2(X, E) = \overline{\bigoplus_{k \geq 1} V_{\lambda_k}}.$$

This is the generalization of the eigenspace decomposition of a self-adjoint operator. We note here that, in general, the above decomposition is not a sum of mutually orthogonal subspaces.

Definition A.9. We call algebraic multiplicity $m(\lambda_k)$ of the eigenvalue λ_k the dimension of the corresponding eigenspace V_{λ_k} .

By equation (A.2), we see that for $\text{Re}(z) > d/m$, D^{-z} is a trace class operator. Then, we can employ the Lidskii's theorem ([Sim05, Theorem 3.7]) to write the ζ -function as

$$\zeta_{\theta}(z, D) = \sum_{k=1}^{\infty} m(\lambda_k) (\lambda_k)_{\theta}^{-z} = \sum_{k=1}^{\infty} m(\lambda_k) e^{-z \log_{\theta} \lambda_k},$$

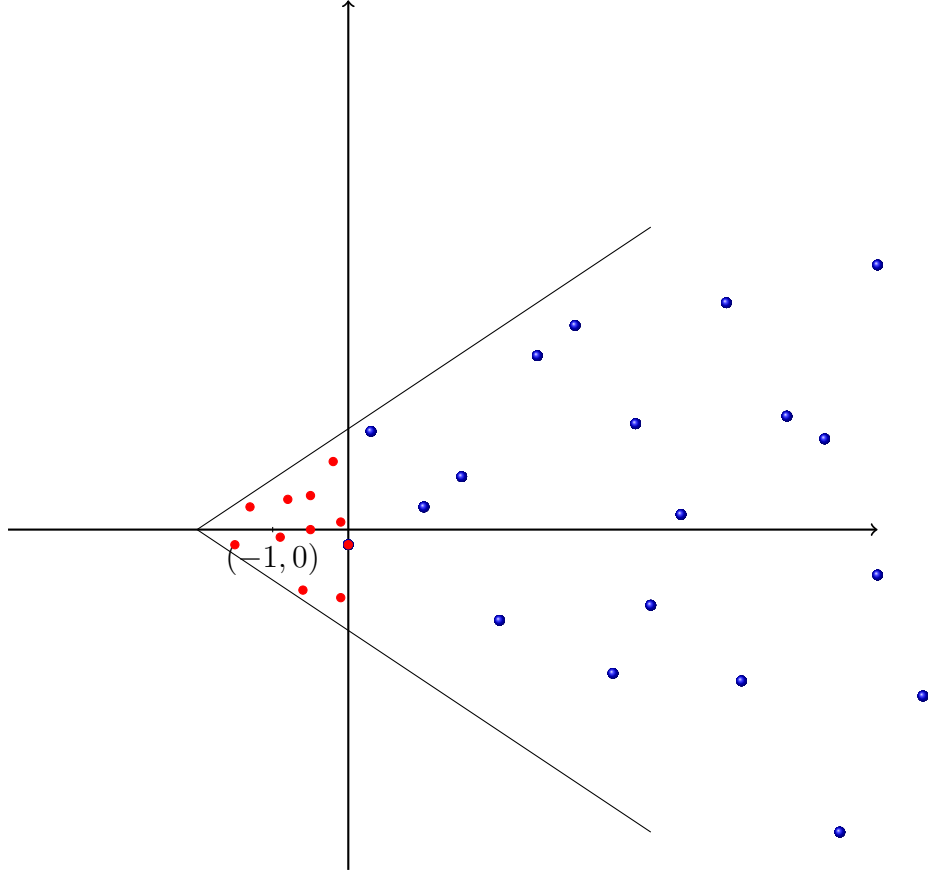


Figure A.1: The discrete spectrum of the operator $(D_\chi^\sharp(\sigma))^2$.

where $\log_\theta \lambda$ denotes the branch of the logarithm in $\mathbb{C} - R_\theta$ with $\theta < \text{Im}(\log_\theta \lambda) < \theta + 2\pi$.

We study now the spectral properties of the twisted Dirac operator $(D_\chi^\sharp(\sigma))^2$. The following lemma describes its spectrum.

Lemma A.10. *Let ε be an angle such that the principal symbol $\sigma_{(D_\chi^\sharp(\sigma))^2}(x, \xi)$ of $(D_\chi^\sharp(\sigma))^2$, for $\xi \in T_x^*X, \xi \neq 0$ does not take values in $L_{[-\varepsilon, \varepsilon]}$. Then, the spectrum $\text{spec}((D_\chi^\sharp(\sigma))^2)$ of the twisted Dirac operator $(D_\chi^\sharp(\sigma))^2$ is discrete and for every ε there exists $R > 0$ such that $\text{spec}((D_\chi^\sharp(\sigma))^2)$ is contained in the set $B(-1, R) \cup L_{[-\varepsilon, \varepsilon]} \subset \mathbb{C}$.*

Proof. The discreteness of the spectrum follows from [Shu87, Theorem 8.4] and for the second statement see [Shu87, Theorem 9.3]. \square

Let θ be an Agmon angle for the operator $(D_\chi^\sharp(\sigma))^2$. Then, by definition of the

Agmon angle and Lemma A.10, there exists $\varepsilon > 0$ such that

$$\text{spec}((D_\chi^\sharp(\sigma))^2) \cap L_{[\theta-\varepsilon, \theta+\varepsilon]} = \emptyset.$$

Since $(D_\chi^\sharp(\sigma))^2$ has discrete spectrum, there exists also an $r_0 > 0$ such that

$$\text{spec}((D_\chi^\sharp(\sigma))^2) \cap \{z \in \mathbb{C} : |z + 1| \leq 2r_0\} = \emptyset.$$

We define a contour Γ_{θ, r_0} as follows.

$$\Gamma_{\theta, r_0} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3,$$

where $\Gamma_1 = \{-1 + re^{i\theta} : \infty > r \geq r_0\}$, $\Gamma_2 = \{-1 + r_0e^{ia} : \theta \leq a \leq \theta + 2\pi\}$, $\Gamma_3 = \{-1 + re^{i(\theta+2\pi)} : r_0 \leq r < \infty\}$. On Γ_1 , r runs from ∞ to r_0 , Γ_2 is oriented counterclockwise, and on Γ_3 , r runs from r_0 to ∞ .

We put

$$e^{-t(D_\chi^\sharp(\sigma))^2} = \frac{i}{2\pi} \int_{\Gamma_{\theta, r_0}} e^{-t\lambda} ((D_\chi^\sharp(\sigma))^2 - \lambda \text{Id})^{-1} d\lambda \quad (\text{A.4})$$

$$D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2} = \frac{i}{2\pi} \int_{\Gamma_{\theta, r_0}} \lambda^{1/2} e^{-t\lambda} ((D_\chi^\sharp(\sigma))^2 - \lambda \text{Id})^{-1} d\lambda \quad (\text{A.5})$$

We have $|e^{-t\lambda}| \leq e^{-t\text{Re}(\lambda)}$. Furthermore by [Shu87, Corollary 9.2], there exist a positive constant $c > 0$ such that $\|((D_\chi^\sharp(\sigma))^2 - \lambda \text{Id})^{-1}\| \leq c|\lambda|^{-1}$. Hence, the integrals in (A.4) and (A.5) are well defined.

In Section 4.3, we define the operator $A_\chi^\sharp(\sigma)$ acting on $C^\infty(X, E_\tau \otimes E_\chi)$. Since it is induced by the twisted Bochner-Laplace operator $\Delta_{\tau, \chi}^\sharp$, it is a second order elliptic differential operator, which has nice spectral properties. We have the following lemma.

Lemma A.11. *Let ε be an angle such that the principal symbol $\sigma_{A_\chi^\sharp(\sigma)}(x, \xi)$ of $A_\chi^\sharp(\sigma)$, for $\xi \in T_x^*X, \xi \neq 0$ does not take values in $L_{[-\varepsilon, \varepsilon]}$. Then, the spectrum $\text{spec}(A_\chi^\sharp(\sigma))$ of $A_\chi^\sharp(\sigma)$ is discrete and for every ε there exists $R > 0$ such that $\text{spec}(A_\chi^\sharp(\sigma))$ is contained in the set $B(-1, R) \cup L_{[-\varepsilon, \varepsilon]} \subset \mathbb{C}$.*

Proof. As in the proof of Lemma A.10. □

Given an Agmon angle θ for the operator $A_\chi^\sharp(\sigma)$ and $r_0 > 0$ we can consider a contour Γ_{θ, r_0} in the same way as for the operator $(D_\chi^\sharp(\sigma))^2$. Then, we put

$$e^{-tA_\chi^\sharp(\sigma)} = \frac{i}{2\pi} \int_{\Gamma_{\theta, r_0}} e^{-t\lambda} (A_\chi^\sharp(\sigma) - \lambda \text{Id})^{-1} d\lambda. \quad (\text{A.6})$$

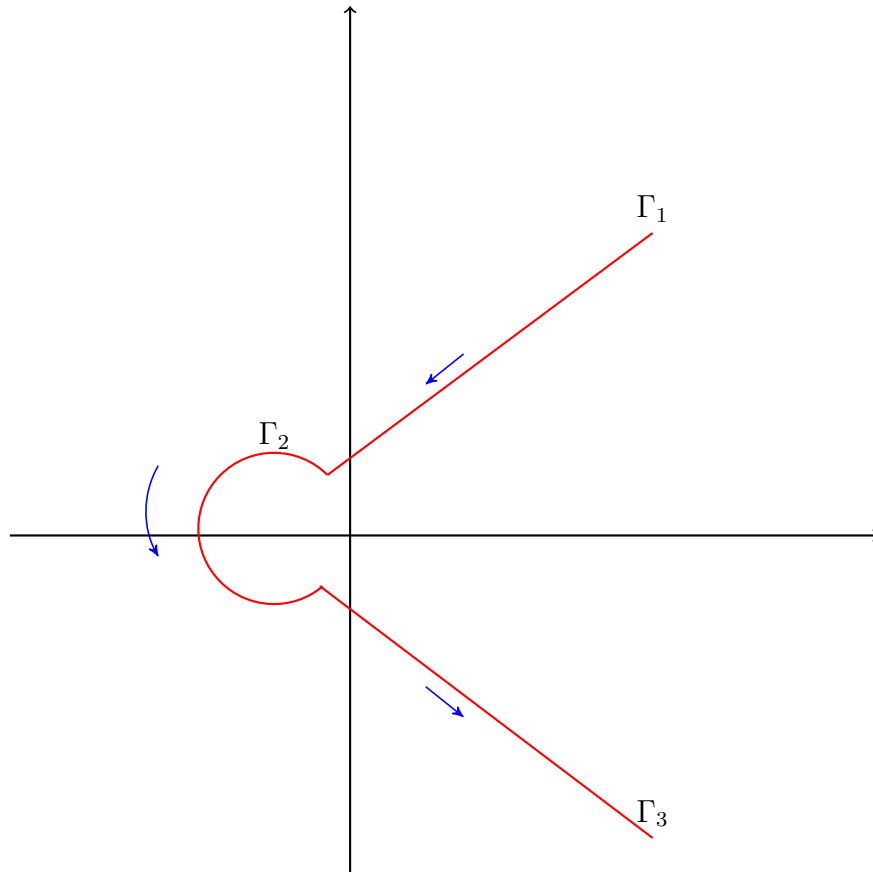


Figure A.2: The contour $\Gamma_{\theta, r_0} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$.

By [Shu87, Corollary 9.2] and the fact that $|e^{-t\lambda}| \leq e^{-t\operatorname{Re}(\lambda)}$, the integral in equation (A.6) is well defined.

The kernel of the integral operator $D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2}$ is given by

$$K_t^{\tau_s(\sigma),\chi}(x, x') = \sum_{\gamma \in \Gamma} K_t^{\tau_s(\sigma)}(g^{-1}\gamma g') \otimes \chi(\gamma), \quad (\text{A.7})$$

where $x = \Gamma g$, $x' = \Gamma g'$, $g, g' \in G$, as in Section 5.2.

In Section 5.3, we used the asymptotic expansion of the trace of the kernel $K_t^{\tau_s(\sigma),\chi}(x, y)$, which is described by the following lemma.

Lemma A.12. *The asymptotic expansion of the trace of the kernel $K_t^{\tau_s(\sigma),\chi}(x, y)$ of the operator $D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2}$ is given by*

$$\operatorname{tr} K_t^{\tau_s(\sigma),\chi}(x, y) \sim_{t \rightarrow 0^+} \dim(V_\chi)(a_0(x)t^{1/2} + O(t^{3/2}, x)), \quad (\text{A.8})$$

where $a_0(x)$ is a C^∞ -function on X .

Proof. By [BF86, Theorem 2.4] we have that the trace of the kernel $K_t^{\tau_s(\sigma)}(x, y) \in C(X, E_{\tau_s(\sigma)} \boxtimes E_{\tau_s(\sigma)}^*)$, associated to the integral operator $D(\sigma)e^{-tD^2(\sigma)}$ has the asymptotic expansion

$$\operatorname{tr} K_t^{\tau_s(\sigma)}(x, x) \sim_{t \rightarrow 0^+} a_0(x)t^{1/2} + O(t^{3/2}, x),$$

where $a_0(x)$ a smooth local invariant determined by the total symbol of $D(\sigma)$. Locally, the twisted Dirac operator $D_\chi^\sharp(\sigma)$ is described by (A.1):

$$\tilde{D}_\chi^\sharp(\sigma) = \tilde{D}(\sigma) \otimes \operatorname{Id}_{V_\chi},$$

and the symbol $\sigma_{D_\chi^\sharp(\sigma)}$ of $D_\chi^\sharp(\sigma)$ for $\xi \in T^*X, \xi \neq 0$ is given by

$$\sigma_{D_\chi^\sharp(\sigma)}(x, \xi) = (i\xi) \otimes \operatorname{Id}_{(V_{\tau(\sigma)} \otimes V_\chi)_x}.$$

Hence, by (A.7)

$$\operatorname{tr} K_t^{\tau_s(\sigma),\chi}(x, y) \sim_{t \rightarrow 0^+} \dim(V_\chi)(a_0(x)t^{1/2} + O(t^{3/2}, x)).$$

□

We want to write the asymptotic expansion of the trace of the kernel $H_t^{\tau,\chi}(x, x')$ of the integral operator $e^{-tA_\chi^\sharp(\sigma)}$:

$$H_t^{\tau,\chi}(x, x') = \sum_{\gamma \in \Gamma} e^{-tc(\sigma)} H_t^\tau(g^{-1}\gamma g') \otimes \chi(\gamma), \quad (\text{A.9})$$

where $x = \Gamma g$, $x' = \Gamma g'$, $g, g' \in G$, as in Section 4.3. In Chapter 8, we used the asymptotic expansion of the heat kernel $H_t^{\tau,\chi}(x, x')$.

Lemma A.13. *The asymptotic expansion of the trace of the kernel $H_t^{\tau, \chi}(x, y)$ of the operator $e^{-tA_\chi^\sharp(\sigma)}$ is given by*

$$\mathrm{tr} H_t^{\tau, \chi}(x, x) \sim_{t \rightarrow 0^+} \dim(V_\chi) \sum_{j=0}^{\infty} c_j(x) t^{j - \frac{d}{2}} \quad (\text{A.10})$$

where $c_j(x)$ are C^∞ -functions on X .

Proof. By [Gil95, Lemma 1.8.2] we have that the trace of the kernel $H_t^\tau(x, y) \in C(X, E_\tau \boxtimes E_\tau^*)$, associated to the operator e^{-tA_τ} has the asymptotic expansion

$$\mathrm{tr} H_t^\tau(x, x) \sim_{t \rightarrow 0^+} \sum_{j=0}^{\infty} c_j(x) t^{j - \frac{d}{2}}.$$

where $c_j(x)$ are smooth local invariants determined by the total symbol of the Δ_τ . We recall here that the operator A_τ is just the Bochner-Laplace operator Δ_τ minus the Casimir eigenvalue λ_τ , i.e.,

$$\tilde{A}_\tau = \tilde{\Delta}_\tau - \lambda_\tau,$$

acting on $C^\infty(X, E_\tau)$. On the other hand, locally, the twisted Bochner-Laplace operator $\Delta_{\tau, \chi}^\sharp$ is described as follows:

$$\tilde{\Delta}_{\tau, \chi}^\sharp = \tilde{\Delta}_\tau \otimes \mathrm{Id}_{V_\chi}.$$

Hence, its symbol is given by

$$\sigma_{\Delta_{\tau, \chi}^\sharp}(x, \xi) = (\|\xi\|^2) \otimes \mathrm{Id}_{(V_\tau \otimes V_\chi)_x},$$

for $\xi \in T^*X, \xi \neq 0$.

Moreover, if j is an odd integer, then $c_j = 0$. We mention here that we can use the expansion in power series of the term $e^{-tc(\sigma)}$. Then, the assertion follows from equation (A.9). \square

APPENDIX B

The heat equation

We are interested in integral operators, and namely the heat operator $e^{-t\Delta}$, where Δ is the Laplacian applied to smooth sections of a vector bundle over a compact manifold X .

Let X be a compact riemannian manifold, without boundary, of dimension d . Following [Mül12a], we let $\pi : E \rightarrow X$ be a Hermitian vector bundle over X with a fiber metric h . Let $\langle \cdot, \cdot \rangle$ be an inner product in the space $C^\infty(X, E)$ of smooth sections of E , defined by the riemannian metric g on X and the fiber metric h , as follows

$$\langle \phi, \psi \rangle := \int_X h(\phi(x), \psi(x)) dx, \quad \phi, \psi \in C^\infty(X, E), \quad (\text{B.1})$$

where $dx := d\text{Vol}(x)$ denotes the volume form with respect to g . Let $\|\cdot\|$ be the norm in $C^\infty(X, E)$ induced by the inner product (B.1). Let

$$L^2(X, E) := \overline{C^\infty(X, E)}^{\|\cdot\|}.$$

We denote by Δ a Laplace-type operator, that is an elliptic differential operator of order 2, which is

1. formally self-adjoint: $\langle \Delta\phi, \psi \rangle = \langle \phi, \Delta\psi \rangle, \quad \forall \phi, \psi \in C^\infty(X, E)$;
2. positive: $\langle \Delta\phi, \phi \rangle \geq 0, \phi \in C^\infty(X, E)$.

Since X is a closed manifold, we know that $\Delta : C^\infty(X, E) \rightarrow L^2(X, E)$ is essentially self-adjoint, i.e. it admits a unique self adjoint extension, which we will also denote by Δ . Then

$$\Delta : \text{Dom}(\Delta) \rightarrow L^2(X, E),$$

where $Dom(\Delta) = H^2(X, E)$ is the Sobolev space of order 2.

Lemma B.1. *Let $\Delta : Dom(\Delta) \rightarrow L^2(X, E)$ be a Laplace-type operator of order 2. Then, there exists an orthonormal base $(\phi_i), i \in \mathbb{N}$ of $L^2(X, E)$ such that*

1. $\phi_i \in Dom(\Delta)$ and $\Delta\phi_i = \lambda_i\phi_i, \forall i \in \mathbb{N}$
2. $spec(\Delta) = \{0 \leq \lambda_1 \leq \lambda_2 \leq \dots \rightarrow +\infty\}$, and $+\infty$ is the only point of accumulation.

Proof. This is proved in [Gil95, Lemma 1.6.3]. □

The Laplace-type operator Δ generates a semi-group $\{e^{-t\Delta} : t \geq 0\}$. Let $\phi \in L^2(X, E)$. The heat operator $e^{-t\Delta}\phi$ is defined by the spectral decomposition by

$$e^{-t\Delta}\phi := \sum_{i=1}^{\infty} e^{-t\lambda_i} \langle \phi_i, \phi \rangle \phi_i, \quad (\text{B.2})$$

where $(\phi_i), i \in \mathbb{N}$ as in Lemma B.1. The series in (B.2) converges in $L^2(X, E)$.

We consider now the initial-value problem

$$\begin{cases} (\frac{\partial}{\partial t} + \Delta)u^\phi(t; x) = 0 & u^\phi(t; \cdot) \in Dom(\Delta) \\ \lim_{t \rightarrow 0} u^\phi(t; x) = \phi(x) \end{cases} \quad (\text{B.3})$$

where $\phi(x) \in C^\infty(X, E)$.

We want to prove that

$$e^{-t\Delta}\phi(x) = u^\phi(t; x),$$

i.e. $e^{-t\Delta}\phi(x)$ is the unique solution of the initial value problem (B.3), for $t \geq 0$. This is a consequence of the analytic properties of the heat operator.

Lemma B.2. *The heat operator $e^{-t\Delta}$ is a bounded operator.*

Proof. We have

$$\|e^{-t\Delta}\phi\|^2 = \sum_{j=1}^{\infty} e^{-2t\lambda_j} |\langle \phi_j, \phi \rangle|^2 \leq \sum_{j=1}^{\infty} |\langle \phi_j, \phi \rangle|^2 = \|\phi\|^2.$$

□

Lemma B.3. *Let $\phi \in Dom = H^2(X, E)$. Then $\forall t \geq 0$*

$$e^{-t\Delta}\phi - \phi = - \int_0^t e^{-s\Delta} \Delta\phi ds.$$

Proof. We have by the positivity of Δ and (B.2)

$$\begin{aligned}
e^{-t\Delta}\phi - \phi &= \sum_{i=1}^{\infty} (e^{-t\lambda_i} - \text{Id}) \langle \phi_i, \phi \rangle \phi_i = \sum_{i=1}^{\infty} \lambda_i \int_0^t e^{-s\lambda_i} ds \langle \phi_i, \phi \rangle \phi_i \\
&= - \int_0^t \sum_{i=1}^{\infty} e^{-s\lambda_i} \langle \lambda_i \phi_i, \phi \rangle \phi_i ds = - \int_0^t \sum_{i=1}^{\infty} e^{-s\lambda_i} \langle \Delta \phi_i, \phi \rangle \phi_i ds \\
&= - \int_0^t \sum_{i=1}^{\infty} e^{-s\lambda_i} \langle \phi_i, \Delta \phi \rangle \phi_i ds \\
&= - \int_0^t e^{-s\Delta} \Delta \phi ds.
\end{aligned}$$

□

Theorem B.4. Let $\phi(x) \in H^2(X, E)$ and let $u^\phi(t; x) := e^{-t\Delta}\phi(x)$, for $t \in \mathbb{R}^+$. Then, $u^\phi(t; x)$ is the unique solution of the heat equation (B.3).

Proof. By Lemma B.3 above we have that the function $\mathbb{R}^+ \ni t \rightarrow u^\phi(t; x) \in \text{Dom}(\Delta)$ is differentiable and furthermore

$$\begin{aligned}
\frac{d}{dt} u^\phi(t; x) &= -\Delta e^{-t\Delta} \phi(x) = -\Delta u(t; x), \\
\lim_{t \rightarrow \infty} u^\phi(t; x) &= \phi(x).
\end{aligned}$$

□

Lemma B.5. For every $k \in \mathbb{N}$ and for every $t > 0$ we have

$$\sum_{j=1}^{\infty} \lambda_j^k e^{-t\lambda_j} < \infty. \tag{B.4}$$

Proof. Let $\lambda \geq 0$. We define the counting function $N(\lambda; \Delta)$ of the eigenvalues of Δ as

$$N(\lambda; \Delta) := \#\{j : \lambda_j \leq \lambda\}. \tag{B.5}$$

By a rough estimation, we can see that $N(\lambda; \Delta)$ grows at most polynomially in λ , i.e., there exist $C > 0$ and $n \in \mathbb{N}$, such that

$$N(\lambda; \Delta) \leq C(1 + \lambda^n). \tag{B.6}$$

Hence, for every $t > 0$, we have

$$\begin{aligned}
\sum_{i=1}^{\infty} \lambda_i^k e^{-t\lambda_i} &< \sum_{i=1}^{\infty} \sum_{\lambda_i \leq \lambda \leq \lambda_{i+1}} \lambda_i^k e^{-t\lambda_i} < \sum_{i=1}^{\infty} N(i+1; \Delta) (i+1)^k e^{-t(i+1)} \\
&\leq C \sum_{i=1}^{\infty} (1 + (i+1)^n) (i+1)^k e^{-t(i+1)} < \infty.
\end{aligned}$$

□

Remark B.6. *The estimate (B.6) is a rather rough estimate for the counting function $N(\lambda; \Delta)$ compared to the Weyl law, which is obtained using the asymptotic expansion of the heat kernel of the heat operator $e^{-t\Delta}$, for the elliptic operator Δ , as $t \rightarrow 0^+$ ([Gil95, Lemma 1.7.4]) and the Karamata Theorem ([GV92, Theorem 2.42]). We have*

$$N(\lambda; \Delta) = \frac{\text{rk}(E) \text{Vol}(X)}{(4\pi)^{d/2} \Gamma(d/2 + 1)} \lambda^{d/2} + o(\lambda^{d/2}), \quad \lambda \rightarrow \infty,$$

where $\text{rk}(E)$ denotes the rank of the vector bundle E , and $\Gamma(d/2 + 1)$ is the Gamma function.

Lemma B.7. *For every $k \in \mathbb{N}$ the operator $\Delta^k e^{-t\Delta}$ is a bounded operator in $L^2(X, E)$.*

Proof. Let $\phi \in L^2(X, E)$. From the spectral resolution of Δ and Lemma B.5 above, it follows that

$$\|\Delta^k e^{-t\Delta}\|^2 \leq \sum_{i=1}^{\infty} \lambda_i^{2k} e^{-2t\lambda_i} \|\phi\|^2 < \infty.$$

□

We obtain the following theorem.

Theorem B.8. *For all $t > 0$, $e^{-t\Delta}$ is a smoothing operator.*

Proof. It is sufficient to show that $e^{-t\Delta}$ admits an extension to a linear, bounded operator

$$e^{-t\Delta} : H^{2l}(X, E) \rightarrow H^{2(k+l)}(X, E),$$

for all $l \in \mathbb{Z}$, and $k \in \mathbb{N}$. Let $k \in \mathbb{N}$. Then $(\text{Id} + \Delta)^k$ is an elliptic, differential operator of order $2k$. From elliptic regularity theory, we have that there exist positive constants $C_1, C_2 > 0$, such that for every $s \in \mathbb{R}$, and for every $\phi \in H^2(X, E)$

$$C_1 \|(\text{Id} + \Delta)^k \phi\|_s \leq \|\phi\|_{2k+s} \leq \|(\text{Id} + \Delta)^k \phi\|_s. \quad (\text{B.7})$$

Let $l \in \mathbb{Z}$ and $k \in \mathbb{N}$. Using (B.6), and Lemma B.7, we get

$$\begin{aligned} \|e^{-t\Delta} \phi\|_{2(k+l)} &\leq C_2 \|(\text{Id} + \Delta)^{l+k} e^{-t\Delta} \phi\|_0 \\ &\leq C_2 \|(\text{Id} + \Delta)^k e^{-t\Delta} (\text{Id} + \Delta)^l \phi\|_0 \\ &\leq C_2 \|(\text{Id} + \Delta)^k e^{-t\Delta}\| \|(\text{Id} + \Delta)^l \phi\|_0 \\ &\leq C_3 \|\phi\|_{2l}. \end{aligned}$$

The assertion follows. □

Remark B.9. By Theorem B.8 we have

$$e^{-t\Delta}(L^2(X, E)) \subset C^\infty(X, E).$$

Hence, we get a better version of Theorem B.4, namely given $\phi \in C^\infty(X, E)$, then the function $u^\phi(t; x) := e^{-t\Delta}\phi(x)$ is a solution of the initial-valued problem (B.3), i.e.

1. $u^\phi(t; x) \in C^\infty(\mathbb{R}^+ \times X, E)$ and $(\frac{\partial}{\partial t} + \Delta)u^\phi(t; x) = 0$
2. $\lim_{t \rightarrow 0^+} u^\phi(t; x) = \phi(x)$.

We want to describe now the kernel $H(t; x, y)$ of the integral operator $e^{-t\Delta}$ locally.

We observe at first that for $(t, x, y) \in \mathbb{R}^+ \times X \times X$,

$$H(t; x, y) \in \text{Hom}(E_y, E_x) \cong E_x \otimes E_y^*.$$

By definition of the kernel we have for $\phi \in L^2(X, E)$,

$$e^{-t\Delta}\phi(x) = \int_X H(t; x, y)\phi(y)dy. \quad (\text{B.8})$$

Theorem B.10. Let $t \in \mathbb{R}^+$, and $\lambda_i \in \text{spec}(\Delta)$. Let $(\phi_i), i \in \mathbb{N}$ be an orthonormal base of $L^2(X, E)$, consisting of eigenfunctions of Δ . Then the kernel $H(t; x, y)$ of the heat operator has the following expansion

$$H(t; x, y) = \sum_{i=1}^{\infty} e^{-t\lambda_i} \phi_i(x) \otimes \phi_i^*(y). \quad (\text{B.9})$$

The series converges in the C^∞ -topology.

Proof. For $t > 0$, let

$$\tilde{H}(t; x, y) = \sum_{j=1}^{\infty} e^{-t\lambda_j} \phi_j(x) \otimes \phi_j^*(y).$$

Let $r \geq 0$ and let $s \in \mathbb{N}$ such that $2s > d/2 + r$. By the Sobolev embedding theorem and the elliptic regularity theorem ([LM89]) we have that there exist positive constants $C_1, C_2 > 0$ such that

$$\|\phi_i\|_{C^r} \leq C_1 \|\phi_i\|_{2s} \leq C_2 (\|\phi_i\|_0 + \|\Delta^s \phi_i\|_0) = 1 + \lambda_i^s. \quad (\text{B.10})$$

Hence,

$$\|\tilde{H}(t; x, y)\|_{C^r} \leq C_2^2 \sum_{i=1}^{\infty} e^{-t\lambda_i} (1 + \lambda_i^s) < \infty.$$

Therefore, it follows that the series in (B.9) converges in the C^∞ -topology, and that $\tilde{H}(t; x, y)$ is a C^∞ -section of $E \boxtimes E^*$.

Furthermore, we have for $\phi \in L^2(X, E)$,

$$e^{-t\Delta}\phi(x) = \sum_{i=1}^{\infty} e^{-t\lambda_i} \langle \phi_i, \phi \rangle \phi_i(x) = \int_X \tilde{H}(t; x, y) \phi(y) dy.$$

Since the kernel of the heat operator $e^{-t\Delta}$ is uniquely determined, it follows that $\tilde{H} = H$. \square

Corollary B.11. *For every $t > 0$, the operator $e^{-t\Delta}$ is an integral operator with a smooth kernel $H \in C^\infty(X \times X, \text{End}(E))$.*

Proof. This follows from Theorems B.8 and B.10. \square

Remark B.12. *By the analytic properties of the heat operator and Remark 10 we have*

1. $(\frac{\partial}{\partial t} + \Delta) e^{-t\Delta} = 0$
2. $\forall \phi \in L^2(X, E) : \lim_{t \rightarrow 0^+} e^{-t\Delta} \phi = \phi$.
3. $(e^{-t\Delta})^* = e^{-t\Delta}$.

Hence, the kernel $H(t; x, y)$ of the heat operator has the following properties

1. $(\frac{\partial}{\partial t} + \Delta) H(t; x, y) = 0$
2. $\forall \phi \in L^2(X, E) : \lim_{t \rightarrow 0^+} \int_X H(t; x, y) \phi(y) dy = \phi(x)$.
3. $H(t; x, y)^* = H(t; x, y)$.

It can be proved that the heat kernel $H(t; x, y)$ is uniquely determined by 1-3 ([Mül12a, Proposition 1.8]). We call the heat kernel $H(t; x, y)$ the “fundamental solution of the heat equation”.

Theorem B.13. *For every $t > 0$, $e^{-t\Delta}$ is a trace class operator.*

Proof. Let $(\phi_i), i \in \mathbb{N}$ be an orthonormal base of $L^2(X, E)$, consisting of eigenfunctions of Δ . Since the Hilbert-Schmidt norm $\|\cdot\|_{HS}$ is independent of the choice of the orthonormal base we have

$$\begin{aligned} \|e^{-t\Delta}\|_{HS}^2 &= \sum_{i=1}^{\infty} \|e^{-t\Delta}\phi_i\|^2 = \sum_{i=1}^{\infty} e^{-2t\lambda_i} \\ &< \sum_{i=1}^{\infty} \sum_{\lambda_i \leq \lambda \leq \lambda_{i+1}} e^{-t\lambda} < \sum_{i=1}^{\infty} N(i+1; \Delta) e^{-t(i+1)} \\ &\leq C \sum_{i=1}^{\infty} (1 + (i+1)^n) e^{-t(i+1)} < \infty, \end{aligned}$$

where we used the estimate (B.6) for the counting function $N(\lambda; \Delta)$. Hence, $e^{-t\Delta}$ is a Hilbert-Schmidt operator. By the semigroup property we have

$$e^{-2t\Delta} = e^{-t\Delta} \circ e^{-t\Delta}.$$

Therefore, $e^{-2t\Delta}$ is of trace class. \square

We want now to compute the trace of the heat operator. Since $H(t; x, x) \in \text{End}(E_x)$, its trace is a functional on endomorphisms, $\text{tr} H(t; x, x) : \text{End}(E_x) \rightarrow \mathbb{C}$. We have by (B.9)

$$\text{tr} H(t; x, x) = \sum_{i=1}^{\infty} e^{-t\lambda_i} \|\phi_i\|^2.$$

Integrating over X , we get

$$\int_X \text{tr} H(t; x, x) dx = \sum_{i=1}^{\infty} e^{-t\lambda_i}.$$

We have

$$\text{Tr}(e^{-t\Delta}) := \sum_{i=1}^{\infty} \langle e^{-t\Delta} \phi_i, \phi_i \rangle = \sum_{i=1}^{\infty} e^{-t\lambda_i}.$$

All in all, we have proved the following proposition.

Proposition B.14. *Let $H \in C^\infty(X \times X, \text{End}(E))$ be the kernel of the heat operator $e^{-t\Delta}$. Then, for all $t > 0$ we have*

$$\text{Tr}(e^{-t\Delta}) = \int_X \text{tr} H(t; x, x) dx. \quad (\text{B.11})$$

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