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Research Article

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A prime geodesic theorem for $\mathrm{SL}_3(\mathbb{Z})$

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Abstract: We show a prime geodesic theorem for the group $\mathrm{SL}_3(\mathbb{Z})$ counting those geodesics whose lifts lie in the split Cartan subgroup. This is the first arithmetic prime geodesic theorem of higher rank for a non-co-compact group.

Keywords: Prime geodesic theorem, class numbers

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Introduction

In this paper, we show a prime geodesic theorem for congruence subgroups of $\mathrm{SL}_3(\mathbb{Z})$. This is the first example of a prime geodesic theorem for a non-cocompact arithmetic group and geodesics in a Cartan subgroup of split-rank bigger than one. The case of split-rank one has been considered in [10, 16]. Higher-rank cases in a cocompact setting have been considered in [6, 8, 12, 13] and for p-adic groups in [11].

We now present the prime geodesic theorem for $\mathrm{SL}_3(\mathbb{Z})$ in a formulation involving class numbers as follows: Let $O_{\mathbb{R}}(3)$ denote the set of all orders $\mathcal{O}$ in totally real number fields F of degree 3. For such an order $\mathcal{O}$, let $h(\mathcal{O})$ be its class number, $R(\mathcal{O})$ its regulator. For $\lambda \in \mathcal{O}^\times$, let ρ_1, ρ_2, ρ_3 denote the real embeddings of F ordered in a way that $|\rho_1(\lambda)| \geq |\rho_2(\lambda)| \geq |\rho_3(\lambda)|$. Let

$$\alpha_1(\lambda) = \frac{|\rho_1(\lambda)\rho_3(\lambda)|}{|\rho_2(\lambda)|^2}, \quad \alpha_2(\lambda) = \left(\frac{|\rho_2(\lambda)|}{|\rho_3(\lambda)|}\right)^2.$$

Theorem (Prime geodesic theorem for $\mathrm{SL}_3(\mathbb{Z})$). *For $T_1, T_2 > 0$, set*

$$\vartheta(T) = \sum_{\substack{\mathcal{O} \in O_{\mathbb{R}}(3),\, \lambda \in \mathcal{O}^\times/\pm 1 \\ 1 < \alpha_1(\lambda) \leq T_1 \\ 1 < \alpha_2(\lambda) \leq T_2}} R(\mathcal{O}) h(\mathcal{O}),$$

where $h(\mathcal{O})$ is the class number of $\mathcal{O}$ and $R(\mathcal{O})$ its regulator. Then we have, as $T_1, T_2 \to \infty$,

$$\vartheta(T_1, T_2) \sim \frac{16}{\sqrt{3}} T_1 T_2.$$

In the paper, we further give a conjectural Lefschetz formula for general congruence arithmetic groups which, if proven, would imply a similar prime geodesic theorem for arbitrary congruence groups.

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The proof uses Arthur's trace formula applied to test functions which in the compact case [7] yield a Lefschetz formula. In applications of the trace formula, one usually has to choose between test functions which are constructed geometrically, and one knows the orbital integrals or test functions, which come by some functional calculus and one knows the traces of. The work of Harish-Chandra allows one to compute orbital integrals from traces and traces from orbital integrals through some more or less explicit integral transforms. The general problem, though, is that all these distributions, i.e., traces and orbital integrals, are invariant distributions and, in the non-compact case, the trace formula contains non-invariant distributions. Arthur has seen this problem early on and has formulated the invariant trace formula. In this, however, the invariant distributions in the trace formula are given through an inductive process and are rather inexplicit. This is not a problem in applications which involve comparisons of trace formulae for different groups, and one does not need explicit versions of the distributions as long as one knows they agree for matching test functions on different groups. For applications like the analytic continuation of Selberg-like zeta functions, however, it is necessary to compute the terms in the trace formula more explicitly. Here the non-invariant distributions have resisted all efforts so far. The strategy of the paper therefore, similar to [10], consists in computing the terms that contribute the first singularity of the zeta function (with the largest real part) and then to estimate the remaining terms to show they do not do harm on a suitable half-plane which includes the first singularity. This is a setting which allows for the application of a Tauberian theorem similar to the proof of the prime number theorem.

The present proof only works for SL(3) since it uses several tailor-made techniques to block out unwieldy contributions from the trace formula. One of these is the twisting by characters, which relies on the particularities of the representation theory of SL(3) and completely breaks down in the case SL(4). So the hopes of generalization are with the conjectural Lefschetz formula, which is a distilled version of the information the trace formula yields. It circumvents all mentioned difficulties and, if proven, would allow for a complete analytic continuation and a functional equation of the zeta function.

1 Split groups

Traces

Let G be a connected semisimple Lie group with finite center. Throughout, we assume that G is *split* over the reals, i.e., there exists a split Cartan subgroup H_{sp}. Write $\mathfrak{g}$ for the Lie algebra of G and $\mathfrak{g}_{\mathbb{C}}$ for the complexification of $\mathfrak{g}$. Let $K \subset G$ be a maximal compact subgroup with Cartan involution θ, and let $P_{\mathrm{sp}} = M_{\mathrm{sp}}A_{\mathrm{sp}}N_{\mathrm{sp}}$ be a minimal parabolic subgroup such that $A_{\mathrm{sp}}M_{\mathrm{sp}}$ is θ-stable. Then A_{sp} is θ-stable, and M_{sp} is the centralizer of A_{sp} in K. Further, as G is split, M_{sp} actually is a finite group.

We will normalize all Haar measures of reductive subgroups of G according to Harish-Chandra's normalization as in [18]. Note that this normalization requires $\mathrm{vol}(U) = 1$ for any maximal compact subgroup U of a reductive group. It further depends on a choice of an invariant form B, i.e., a multiple of the Killing form, which we will consider fixed and only specify later.

We fix an irreducible representation (τ, V_τ) of K. Let E_τ denote the G-homogeneous vector bundle over G/K induced by τ. The smooth sections of the bundle E_τ may be viewed as smooth functions $f\colon G \to V_\tau$ satisfying $f(xk) = \tau(k^{-1})f(x)$. Fix some $0 \neq \alpha \in V_\tau^*$. Then $\alpha \circ f$ is a complex-valued function on G of right K-type τ, and the map $f \mapsto \alpha \circ f$ is a linear isomorphism

$$\Gamma^\infty(E_\tau) \xrightarrow{\cong} C^\infty(G)(*, \tau).$$

Here we use the following notation: The compact group $K \times K$ acts on $C^\infty(G)$ by

$$(k, l)f(x) = f(k^{-1}xl).$$

Accordingly, the space decomposes into K-bitypes

$$C^\infty(G) = \overline{\bigoplus_{\gamma,\tau\in\widehat{K}}} C^\infty(G)(\gamma, \tau).$$

We write $C^\infty(G)(*, \tau)$ for the closure of the sum of all $C^\infty(G)(\gamma, \tau)$, $\gamma \in \widehat{K}$. The group G acts on $C^\infty(G)$ by right translations $R(x)\phi(y) = \phi(yx)$. For $f \in C_c^\infty(G)$ and any unitary representation π of G we write

$$\pi(f) = \int_G f(x)\pi(x)\,dx,$$

where dx denotes a fixed Haar measure on G, which we will normalize later. If f is of left K-type τ, then the operator $R(f)$ preserves the right K-type $C^\infty(G)(*, \tau)$. The restriction of $R(f)$ to this right K-type equals $R(P(f))$, where P is the projection to $C^\infty(G)(*, \tau^*)$, where τ^* is the representation dual to τ. Together this means that we may assume $f \in C_c^\infty(G)(\tau, \tau^*)$.

Definition 1.1. Let $\widehat{G}$ denote the *unitary dual* of G, i.e., the set of all irreducible unitary representations up to unitary equivalence.

Let $\widehat{G}_{\mathrm{adm}}$ denote the set of all irreducible admissible representations up to infinitesimal equivalence. By results of Casselman and Harish-Chandra, every $\pi \in \widehat{G}_{\mathrm{adm}}$ can be realized on a Banach space, and $\widehat{G}$ can be viewed as a subset of $\widehat{G}_{\mathrm{adm}}$.

Lemma 1.2. *Consider the convolution algebra $C_c^\infty(G)(\tau, \tau^*)$, and let C_τ denote its center. Then C_τ equals the set of all $f \in C_c^\infty(G)$ such that $\pi(f) = h_f(\pi)P_{\pi,\tau}$ holds for every $\pi \in \widehat{G}$, where $h_f(\pi)$ is a scalar and $P_{\pi,\tau}$ is the orthogonal projection onto the K-isotype $V_\pi(\tau)$.*

If $f \in C_\tau$, then $\pi(f) = h_f(\pi)P_{\pi,\tau}$ also holds for every $\pi \in \widehat{G}_{\mathrm{adm}}$.

Proof. Let $\pi \in \widehat{G}$. Then $\pi(C_c^\infty(C)(\tau, \tau^*)) = P_{\pi,\tau}\pi(C_c^\infty(G))P_{\pi,\tau}$. Therefore, the action of $C_c^\infty(G)(\tau, \tau^*)$ on $V_\pi(\tau)$ is irreducible; hence its center acts by scalars by the lemma of Schur. The other way round, let $f \in C_c^\infty(G)(\tau, \tau^*)$ be such that $\pi(f) = h_f(\pi)P_{\pi,\tau}$ holds for every $\pi \in \widehat{G}$. Then $\pi(f)$ commutes with $\pi(h)$ for every $h \in C_c^\infty(G)(\tau, \tau^*)$; hence f lies in the center of $C_c^\infty(G)(\tau, \tau^*)$ by the Plancherel theorem. □

For a principal series representation $\pi = \pi_{\sigma,\lambda}$ with $\sigma \in \widehat{M}_{\mathrm{sp}}$ and $\lambda \in \mathfrak{a}^*_{\mathrm{sp},\mathbb{C}}$, the scalar $h_f(\pi) = h_f(\sigma, \lambda)$ can be computed as follows. First we assume that the K-type τ actually occurs in π, for otherwise the scalar will be zero. Pick a norm-one vector p_0 in $V_{\sigma,\lambda}(\tau)$. Then p_0 is a continuous function with $p_0(1) \neq 0$ and, using the Peter–Weyl theorem, one has $\int_K f(xk)p_0(yk)\,dk = f(x)p_0(y)$. With the Iwasawa integral formula, one computes

$$\begin{aligned} h_f(\sigma, \lambda)p_0(1) &= \pi(f)p_0(1) \\ &= \int_G f(x)p_0(x)\,dx = \int_{A_{\mathrm{sp}}N_{\mathrm{sp}}K} f(ank)a^{\lambda+\rho}p_0(k)\,da\,dn\,dk = \int_{A_{\mathrm{sp}}} f^{N_{\mathrm{sp}}}(a)a^{\lambda+\rho}\,da\,p_0(1), \end{aligned}$$

where $f^{N_{\mathrm{sp}}}(a) = \int_{N_{\mathrm{sp}}} f(an)\,dn$. We conclude that the scalar

$$h_f(\sigma, \lambda) = \int_{A_{\mathrm{sp}}} f^{N_{\mathrm{sp}}}(a)a^{\lambda+\rho}\,da$$

is independent of σ. We write it as $h_f(\lambda)$. By the theory of Knapp–Stein intertwining operators, we know that $\pi_{\sigma,\lambda} \cong \pi_{w\sigma,w\lambda}$ for $w \in W = W(G, A_{\mathrm{sp}})$, so $h_f(\lambda) = h_f(w\lambda)$ for $w \in W$.

Proposition 1.3. *The map $\Phi\colon f \mapsto f^{N_{\mathrm{sp}}}(a)a^\rho$ is an isomorphism of topological algebras $C_\tau \xrightarrow{\cong} C_c^\infty(A_{\mathrm{sp}})^W$.*

After taking traces, this proposition also yields a part of the invariant Paley–Wiener theorem of [5]. It can, however, not directly be deduced from this result, as the traces alone do not fix the algebra C_τ.

Proof. Note first that, since $h_{f*g} = h_f h_g$, the map Φ is a homomorphism of convolution algebras. The map is linear and continuous. We only need to show bijectivity, as the open mapping theorem then implies continuity of the inverse. For injectivity, let $f \in \ker(\Phi)$. By the above computation, we then have $\pi(f) = 0$ for every $\pi \in \widehat{G}$, so $f = 0$ by the Plancherel theorem. For surjectivity, we employ the Paley–Wiener theorem for reductive groups by James Arthur [1]; see also [24, 25]. In this paper, the Fourier transform of K-finite functions in $C_c^\infty(G)$ is described in terms of growth estimates and additional relations, the so-called *Arthur–Campoli relations*. The Weyl group invariance is a special case of those. We have to show that all Arthur–Campoli relations are satisfied by any function h which is an A_{sp}-Fourier transform of some $g \in C_c^\infty(A_{\mathrm{sp}})^W$. The Arthur–Campoli relations are of the following form: there are tuples of differential operators $D_1, \dots, D_n$ with constant coefficients on $\mathfrak{a}^*_{\mathrm{sp},\mathbb{C}}$ and $\lambda_1, \dots, \lambda_n \in \mathfrak{a}^*_{\mathrm{sp},\mathbb{C}}$ such that

$$\sum_{k=1}^{n} D_k h_f(\lambda_k) = 0$$

holds for every $f \in C_\tau$. Here the differential operator D_k acts with respect to λ_k. We claim that if such a relation holds for all h_f with $f \in C_\tau$, then it holds for every $h \in C^\infty(\mathfrak{a}^*_{\mathrm{sp},\mathbb{C}})^W$. Let $p \in \mathrm{Pol}(\mathfrak{a}^*_{\mathrm{sp},\mathbb{C}})^W$ be a Weyl-group invariant polynomial, and let z_p be the corresponding differential operator in $\mathfrak{z}$ given by the Harish-Chandra isomorphism. Then one has $h_{z_p f} = p h_f$, so $\sum_{k=1}^n D_k(p(\lambda_k) h_f(\lambda_k)) = 0$. The Casimir element $C \in U(\mathfrak{g})$ induces a differential operator of order 2 on E_τ. For an even Paley–Wiener function ϕ on $\mathbb{C}$, the functional calculus $\phi(\sqrt{-C})$ gives a smoothing operator of finite propagation speed [4], which by G-invariance is given by convolution with some $f \in C_\tau$. Varying ϕ such that its support shrinks to zero, the corresponding eigenvalue functions h_f tend to a non-zero constant locally uniformly with all derivatives. Hence we get $\sum_{k=1}^n D_k(p(\lambda_k)) = 0$ for every Weyl group invariant polynomial p. Another approximation shows that $\sum_{k=1}^n D_k(h(\lambda_k)) = 0$ for every $h \in C^\infty(\mathfrak{a}^*_{\mathrm{sp},\mathbb{C}})^W$ as claimed. □

Definition 1.4. Let B denote a fixed positive multiple of the Killing form on $\mathfrak{g}$ and θ the Cartan involution fixing K pointwise. The form $\langle X, Y\rangle = -B(\theta(X), Y)$ is positive definite on $\mathfrak{g}$ and induces a G-invariant Riemannian metric on G/K. Let $\mathrm{dist}(x, y)$ denote the corresponding distance function, and let $d(g) = \mathrm{dist}(gK, eK)$ for $g \in G$.

Let $U(\mathfrak{g}_\mathbb{C})$ denote the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$. Every element X of $U(\mathfrak{g}_\mathbb{C})$ gives rise to a left-invariant differential operator, written $h \mapsto h * X$, and a right-invariant differential operator, written $h \mapsto X * h$. Recall that, for $p > 0$, the L^p-Schwartz space $\mathcal{C}^p(G)$ is defined as the space of all $h \in C^\infty(G)$ such that, for every $n \in \mathbb{N}$ and $X, Y \in U(\mathfrak{g}_\mathbb{C})$, the seminorm

$$|h|_{p,n,X,Y} = \sup_{g\in G} |X * h * Y(g)|\, \Xi(g)^{-2/p}(1 + d(g))^n$$

is finite. Here Ξ is the basic spherical function

$$\Xi(x) = \int_K \underline{a}(kx)^\rho \, dk,$$

where $\underline{a}(x)$ is the A_{sp}-part of the $A_{\mathrm{sp}} N_{\mathrm{sp}} K$-decomposition of $x \in G$. It suffices for our purposes to know that there are $r_1 > r_2 > 0$ such that $e^{-r_1 d(g)} \le \Xi(g) \le e^{-r_2 d(g)}$. If we complete the space $\mathcal{C}^p(G)$ with respect to the seminorms involving only derivatives up to order k, we obtain a Banach space $\mathcal{C}^p_k(G)$. We write

$$\mathcal{C}^p_k(K\backslash G/K) = \mathcal{C}^p_k(G) \cap C(K\backslash G/K)$$

for the set of K-bi-invariant elements of $\mathcal{C}^p_k(G)$.

Definition 1.5. Let $L^1(G, \Xi)$ denote the L^1 space with respect to the measure $\Xi(x)\,dx$, where dx denotes the Haar measure on G. Further, let $L^1_0(G, \Xi)$ denote the space of K-bi-invariant functions in $L^1(G, \Xi)$.

Let $N \in \mathbb{N}$, and let $\mathcal{C}_N(A_{\mathrm{sp}})^W$ denote the space of all N-times continuously differentiable, W-invariant functions g on A_{sp} such that, for every invariant differential operator D of order $\le N$, one has

$$Dg(\exp(X)) = O(\|X\|^{-N})$$

for some fixed norm on $\mathfrak{a}_{\mathrm{sp}}$.

Proposition 1.6. *The following statements hold:*

(a) *The map $\Phi\colon f \mapsto f^{N_{\mathrm{sp}}}(a)a^\rho$ is an isomorphism of topological algebras $L^1_0(G, \Xi) \xrightarrow{\cong} L^1(A_{\mathrm{sp}})^W$.*

(b) *Let $N \in \mathbb{N}$ be large enough. Then $\mathcal{C}_N(A_{\mathrm{sp}})^W$ is a subset of $L^1(A_{\mathrm{sp}})^W$ and, for every $g \in \mathcal{C}_N(A_{\mathrm{sp}})^W$, its inverse image $f = \Phi^{-1}(g)$ lies in $\mathcal{C}^1_N(G)$.*

Proof. (a) For $f \in L^1_0(G, \Xi)$, we estimate the L^1-norm as

$$\begin{aligned}\|\Phi(f)\| &= \int_{A_{sp}} |f^{N_{sp}}(a)| a^\rho \, da = \int_{A_{sp}} \Big| \int_{N_{sp}} f(an) \, dn \Big| a^\rho \, da \\ &\le \int_{A_{sp}} \int_{N_{sp}} |f(an)| a^\rho \, dn \, da = \int_{A_{sp}} \int_{N_{sp}} \int_K |f(ank)| a^\rho \, dk \, dn \, da \\ &= \int_G |f(x)| \underline{a}(x)^\rho \, dx = \int_K \int_G |f(k^{-1}x)| \underline{a}(x)^\rho \, dx \, dk \\ &= \int_K \int_G |f(x)| \underline{a}(kx)^\rho \, dx \, dk = \int_G |f(x)| \Xi(x) \, dx.\end{aligned}$$

So Φ is a continuous homomorphism of Banach algebras. It is injective for the same reason as in Proposition 1.3. In the above estimate, note that we have equality throughout, if $f \ge 0$. The image of Φ contains $C_c^\infty(A_{sp})^W$ and therefore all monotonous limits of such functions, so it contains all positive functions in $L^1(A_{sp})^W$, and hence Φ is surjective.

(b) As f is bi-invariant under K, in the expression $X * f * Y$ it suffices to consider $X, Y \in U(\mathfrak{a}_{sp} \oplus \mathfrak{n}_{sp})$. The $\mathfrak{n}$-derivatives are killed by the integral in $\Phi(f)(a) = \int_{N_{sp}} f(an) \, dn \, a^\rho$, and the $\mathfrak{a}$-derivatives translate to derivatives of $g = \Phi(f)$. This yields the claim. □

Orbital integrals

For $x \in G$, let G_x denote its centralizer. If x is semisimple, the group G_x is reductive, so we have fixed a Haar measure on G_x. Also, the group G_x is unimodular, so on G/G_x there exists a unique compatible G-invariant Radon measure. For a function f on G and a semisimple element $x \in G$, the corresponding *orbital integral* $\mathcal{O}_x(f)$ is defined by

$$\mathcal{O}_x(f) = \int_{G/G_x} f(yxy^{-1}) \, dy = \int_{G_x \backslash G} f(y^{-1}xy) \, dy$$

whenever the integral exists, which, for instance, is the case if f is continuous and of compact support. Let H be a θ-stable Cartan subgroup of G. Let $\mathfrak{h}_\mathbb{C}$ be its complex Lie algebra, and let $\Phi = \Phi(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ be the set of roots. Let $x \to x^c$ denote the complex conjugation on $\mathfrak{g}_\mathbb{C}$ with respect to the real form $\mathfrak{g}$. Choose an ordering $\Phi^+ \subset \Phi$, and let Φ_I^+ be the set of positive imaginary roots. To any root $\alpha \in \Phi$, let

$$H \to \mathbb{C}^\times, \quad h \mapsto h^\alpha$$

be its character, that is, for $X \in \mathfrak{g}_\alpha$ the root space to α and any $h \in H$, we have $\mathrm{Ad}(h)X = h^\alpha X$. Now put

$$\Delta_I(h) = \prod_{\alpha \in \Phi_I^+} (1 - h^{-\alpha}).$$

Let $H = AT$, where A is the connected split component and T is compact. Choose a parabolic P with split component A, so P has Langlands decomposition $P = MAN$. For $at \in AT = H$, let

$$\begin{aligned}\Delta_+(at) &= |\det((1 - \mathrm{Ad}((at)^{-1}))|_{\mathfrak{g}/\mathfrak{a}\oplus\mathfrak{m}})|^{\frac{1}{2}} \\ &= |\det((1 - \mathrm{Ad}((at)^{-1}))|_{\mathfrak{n}})| a^{\rho_P} \\ &= \Big| \prod_{\alpha \in \Phi^+ \setminus \Phi_I^+} (1 - (at)^{-\alpha}) \Big| a^{\rho_P},\end{aligned}$$

where ρ_P is the half of the sum of the roots in $\Phi(P, A)$, i.e., $a^{2\rho_P} = \det(a|\mathfrak{n})$. We will also write h^{ρ_P} instead of a^{ρ_P}.

For any $h \in H^{\mathrm{reg}} = H \cap G^{\mathrm{reg}}$, let

$$F_f^H(h) = F_f(h) = \Delta_I(h)\Delta_+(h) \int_{G/A} f(xhx^{-1})\,dx.$$

Then, for $h \in H^{\mathrm{reg}}$, one has

$$\mathcal{O}_h(f) = \frac{F_f^H(h)}{h^{\rho_P}\det(1 - h^{-1}|(\mathfrak{g}/\mathfrak{h})^+)},$$

where $(\mathfrak{g}/\mathfrak{h})^+$ is the sum of the root spaces attached to positive roots. There is an extension of this identity to non-regular elements as follows: For $h \in H$, let G_h denote its centralizer in G. Let $\Phi^+(\mathfrak{g}_h, \mathfrak{h})$ be the set of positive roots of $(\mathfrak{g}_h, \mathfrak{h})$. Let

$$\varpi_h = \prod_{\alpha\in\Phi^+(\mathfrak{g}_h,\mathfrak{h})} Y_\alpha,$$

where $Y_\alpha \in \mathfrak{h}$ is the unique element such that $B(Y_\alpha, X) = \alpha(X)$ for every $X \in \mathfrak{h}$ and B is the multiple of the Killing form which we use for our normalizations. Then ϖ_h defines a left invariant differential operator on G.

Lemma 1.7. *For $f \in \mathcal{C}_N^1(G)$ and $h \in H$, the orbital integral $\mathcal{O}_n(f)$ exists, and we have*

$$\mathcal{O}_h(f) = \frac{\varpi_h F_f^H(h)}{h^{\rho_P}\det(1 - h^{-1}|(\mathfrak{g}/\mathfrak{g}_h)^+)}.$$

Proof. This is proven in [17, Section 17]. □

Our aim is to express orbital integrals in terms of traces of representations. By the above lemma, it is enough to express $F_f(h)$ in terms of traces of f when $h \in H^{\mathrm{reg}}$. For this, let $H_1 = A_1T_1$ be another θ-stable Cartan subgroup of G, and let $P_1 = M_1A_1N_1$ be a parabolic with split component A_1. Let $K_1 = K \cap M_1$. Since G is connected, the compact group T_1 is an abelian torus, and its unitary dual $\widehat{T_1}$ is a lattice. The Weyl group $W = W(M_1, T_1)$ acts on $\widehat{T_1}$, and $\widehat{t_1} \in \widehat{T_1}$ is called *regular* if its stabilizer $W(\widehat{t_1})$ in W is trivial. The regular set $\widehat{T_1}^{\mathrm{reg}}$ modulo the action of $W(K_1, T_1) \subset W(M_1, T_1)$ parameterizes the discrete series representations of M_1 (see [23]). For $\widehat{t_1} \in \widehat{T_1}$, Harish-Chandra [19] defined a distribution $\Theta_{\widehat{t_1}}$ on G which happens to be the trace of the discrete series representation $\pi_{\widehat{t_1}}$ attached to $\widehat{t_1}$ when $\widehat{t_1}$ is regular. When $\widehat{t_1}$ is not regular, the distribution $\Theta_{\widehat{t_1}}$ can be expressed as a linear combination of traces as follows. Choose an ordering of the roots of (M_1, T_1), and let Ω be the product of all positive imaginary roots. For any $w \in W$, we have $w\Omega = \varepsilon_I(w)\Omega$ for a homomorphism $\varepsilon_I\colon W \to \{\pm 1\}$. For non-regular $\widehat{t_1} \in \widehat{T_1}$, we get $\Theta_{\widehat{t_1}} = \frac{1}{|W(\widehat{t_1})|}\sum_{w\in W(\widehat{t_1})} \epsilon_I(w)\Theta'_{w,\widehat{t_1}}$, where $\Theta'_{w,\widehat{t_1}}$ is the character of an irreducible representation $\pi_{w,\widehat{t_1}}$, called a limit of discrete series representation. We will write $\pi_{\widehat{t_1}}$ for the virtual representation $\frac{1}{|W(\widehat{t_1})|}\sum_{w\in W_{\widehat{t_1}}} \epsilon_I(w)\pi_{w,\widehat{t_1}}$. Further, let ρ_I be $\frac{1}{2}$ times the sum of positive imaginary roots.

Let $\nu\colon a \mapsto a^\nu$ be a unitary character of A_1. Then $\widehat{h_1} = (\nu, \widehat{t_1})$ is a character of $H_1 = A_1T_1$. Let $\Theta_{\widehat{h_1}}$ be the character of the representation $\pi_{\widehat{h_1}}$ induced parabolically from $(\nu, \pi_{\widehat{t_1}})$. Harish-Chandra has proven the following theorem.

Theorem 1.8. *Let $H_1, \dots, H_r$ be a maximal set of non-conjugate θ-stable Cartan subgroups with split components $A_1, \dots, A_r$. Let $H = H_j$ for some j with split component A. Then, for each j, there exists a continuous function $\Phi_{H|H_j}$ on $H^{\mathrm{reg}} \times \hat{H}_j$ such that, for $h \in H^{\mathrm{reg}}$, it holds*

$$F_f^H(h) = \sum_{j=1}^r \int_{\hat{H}_j} \Phi_{H|H_j}(h, \widehat{h_j}) \operatorname{tr}\pi_{\widehat{h_j}}(f)\,d\widehat{h_j}.$$

Further, $\Phi_{H|H_j} = 0$ unless there is $g \in G$ such that $gAg^{-1} \subset A_j$. Finally, for $H_j = H$, the function can be given explicitly as

$$\Phi_{H|H}(h, \widehat{h}) = \frac{1}{|W(G,H)|}\sum_{w\in W(G,H)} \varepsilon_I(w)\langle \hat{h}, w^{-1}h\rangle h^{w\rho_I - \rho_I},$$

where we recall that, although the modular shift ρ needs not be a character of H, the difference $w\rho_I - \rho_I$ always is.

Proof. See [19] for a proof. □

The split Cartan

We continue to assume that G is a split group, i.e., it possesses a Cartan $H_{\mathrm{sp}} = A_{\mathrm{sp}}T_{\mathrm{sp}}$, which is split and θ-stable. Then T_{sp} is the centralizer of A_{sp} in K; it is a finite group. There are no imaginary roots; therefore, ε_I is trivial, and $\rho_I = 0$. We fix a choice of positive roots according to a parabolic $P_{\mathrm{sp}} = A_{\mathrm{sp}}M_{\mathrm{sp}}N_{\mathrm{sp}}$. We write A_{sp}^+ for the positive Weyl chamber.

Lemma 1.9. *If $A_{\mathrm{sp}}T_{\mathrm{sp}}$ is a split Cartan subgroup, then, for $f \in \mathcal{C}_N^1(K\backslash G/K)$ and $a_0 \in A_{\mathrm{sp}}^+$, $t_0 \in T_{\mathrm{sp}}$, one has*

$$\mathcal{O}_{a_0t_0}(f) = \frac{f^{N_{\mathrm{sp}}}(a_0)}{\det(1-(a_0t_0)^{-1}|\mathfrak{n}_{\mathrm{sp}})},$$

where $f^{N_{\mathrm{sp}}}(a) = \int_{N_{\mathrm{sp}}} f(an)\,dn$.

Proof. Let n be the order of the finite group T_{sp}. Then $t_0^n = 1$ and $a_0^n \in A_{\mathrm{sp}}^+$. Therefore,

$$A_{\mathrm{sp}} \subset G_{a_0t_0} \subset G_{a_0^nt_0^n} = G_{a_0} = A_{\mathrm{sp}}T_{\mathrm{sp}}.$$

So there exists a finite group $F \subset T_{\mathrm{sp}}$ such that $G_{a_0t_0} = A_{\mathrm{sp}}F$. Therefore, using the Iwasawa integral formula, we get

$$\mathcal{O}_{a_0t_0}(f) = \int\limits_{A_{\mathrm{sp}}F\backslash A_{\mathrm{sp}}N_{\mathrm{sp}}K} f((ank)^{-1}a_0t_0ank)\,da\,dn\,dk.$$

The group F, centralizing A_{sp}, normalizes N, and its conjugation action on N preserves the Haar measure. Therefore, we end up with an integral over $(A_{\mathrm{sp}}\backslash A_{\mathrm{sp}}N) \times F\backslash K$. The compact groups K and F are both normalized to volume 1; the integrand being invariant under K, we end up with

$$\mathcal{O}_{a_0t_0}(f) = \int\limits_{N_{\mathrm{sp}}} f(n^{-1}a_0t_0n)\,dn.$$

Now $n^{-1}a_0t_0n = a_0t_0(n^{a_0t_0})^{-1}n$, where $n^{a_0t_0} = (a_0t_0)^{-1}n(a_0t_0)$. The map $n \mapsto (n^{a_0t_0})^{-1}n$ is bijective and has differential determinant equal to $\det(1-(a_0t_0)^{-1}|\mathfrak{n}_{\mathrm{sp}})$. The claim follows. □

2 Orbital integrals for SL(3)

The group $G = \mathrm{SL}_3(\mathbb{R})$ possesses two conjugacy classes of Cartan subgroups, the split Cartan $A_{\mathrm{sp}}T_{\mathrm{sp}}$ and the fundamental Cartan A_1T_1. Here A_{sp} is the group of all diagonal matrices with positive entries, and T_{sp} is the group of all diagonal matrices with entries in $\{\pm1\}$. Further, A_1 is the group of all diagonal matrices of the form $\mathrm{diag}(y, y, y^{-2})$, $y > 0$. Finally, T_1 is the group of all matrices $\begin{pmatrix} k & \\ & 1\end{pmatrix}$, $k \in \mathrm{SO}(2)$.

The fundamental Cartan

Let $\mathrm{SL}_2^{\pm}(\mathbb{R})$ denote the group of all real 2×2 matrices of determinant ±1. We fix the parabolic subgroup $P_1 = M_1BN_1$, where M_1 is the group of all matrices of the form $\begin{pmatrix} g & \\ & \det(g)\end{pmatrix}$ with $g \in \mathrm{SL}_2^{\pm}(\mathbb{R})$ and N_1 is the group of all upper triangular matrices with ones on the diagonal and a zero in the $(1,2)$-position. The Cartan $H_1 = A_1T_1$ lies in P_1. Let $\mathfrak{h}_1$ be the Lie algebra of H_1. Then, for $f \in \mathcal{C}_N^1(G)$, we have

$$F_f^{A_1T_1}(at) = \int\limits_{\widehat{A_{\mathrm{sp}}T_{\mathrm{sp}}}} \Phi_{A_1T_1|A_{\mathrm{sp}}T_{\mathrm{sp}}}(at,\hat{h})\,\mathrm{tr}\,\pi_{\hat{h}}(f)\,d\hat{h} + \int\limits_{\widehat{A_1T_1}} \Phi_{A_1T_1|A_1T_1}(at,\hat{h})\,\mathrm{tr}\,\pi_{\hat{h}}(f)\,d\hat{h}.$$

The second integral equals

$$\sum_{\chi\in\widehat{T}_1}\int\limits_{\mathfrak{a}_1^*} \Phi_{A_1T_1|A_1T_1}(at,(\chi,\mu))\,\mathrm{tr}\,\pi_{\sigma_\chi,i\mu}(f)\,d\mu,$$

where $\sigma_\chi \in \widehat{M}_1$ is the discrete series representation with Harish-Chandra parameter χ. As these do not contain $K \cap M_1$ invariant vectors; the operators $\pi_{\sigma_\chi, i\mu}(f_0)$ are all zero. So, in the case $f = f_0 \in C(K\backslash G/K)$, the second integral vanishes. This means that, for $f_0 \in \mathcal{C}^1_N(K\backslash G/K)$ and $at \in A_1T_1$, we get

$$F^{A_1T_1}_{f_0}(at) = \int_{\mathfrak{a}^*_{\mathrm{sp}}} \Phi(at, i\lambda) h_{f_0}(i\lambda) e^{i\lambda(\log a)}\, d\lambda,$$

where we have used the abbreviation $\Phi(at, i\lambda) = \Phi_{A_1T_1|A_{\mathrm{sp}}T_{\mathrm{sp}}}(at, i\lambda)$.

Definition 2.1. Let $\mathcal{C}_N(A_{\mathrm{sp}})^W_0$ denote the set of all $g \in \mathcal{C}_N(A_{\mathrm{sp}})^W$ which vanish on the walls of the Weyl chambers up to order N.

Lemma 2.2. *For given $m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that, for every $g \in \mathcal{C}_N(A_{\mathrm{sp}})^W_0$ with $f_0 = \Phi^{-1}(g)$, the map $t \mapsto \mathcal{O}_{at}(f_0)$, defined for regular t, extends to an m-times continuously differentiable function in t.*

Proof. We need to compute the orbital integral of at more explicitly. Let again $P_1 = M_1A_1N_1$ be the parabolic with split component A_1. Let $\gamma = at$ be a regular element of A_1T_1. Using the $M_1A_1N_1K$-integral formula, as f_0 is K-bi-invariant, we get

$$\begin{aligned}\mathcal{O}_{at}(f_0) &= \int_{A_1T_1\backslash G} f_0(x^{-1}atx)\,dx = \int_{T_1\backslash M_1N_1} f_0(n^{-1}m^{-1}atmn)\,dn\,dm \\ &= \frac{1}{\det(1 - at|\mathfrak{n}_1)} \int_{T_1\backslash M_1}\int_{N_1} f_0(m^{-1}tman)\,dn\,dm.\end{aligned}$$

Let $M_1^0 \cong SL_2(\mathbb{R})$ be the connected component of M_1, and let t_0 be the matrix $\mathrm{diag}(-1, 1, -1)$, so that $M_1 = M_1^0 \sqcup t_0M_1^0$. We first use K-invariance of f and then the KAK integration formula on the group M_1^0 to get

$$\begin{aligned}\mathcal{O}_{at}(f_0) &= \frac{2}{\det(1 - at|\mathfrak{n}_1)} \int_{T_1\backslash M_1^0}\int_{N_1} f_0(m^{-1}tman)\,dn\,dm \\ &= \frac{2}{\det(1 - at|\mathfrak{n}_1)} \int_{B^+}\int_{N_1} f_0(b^{-1}tban)(b^\alpha - b^{-\alpha})\,dn\,db,\end{aligned}$$

where B^+ is the set of all matrices of the form $\mathrm{diag}(e^s, e^{-s}, 1)$ with $s > 0$ and α is the positive root of (B, M_1). The function $f_{0,a}(x) = \int_{N_1} f_0(xan)\,dn$ is $K_1 = K \cap M_1^0$ bi-invariant on M_1^0. Hence there is a function $\phi_{0,a}$ of one variable such that $f_{0,a}(x) = \phi_{0,a}(\mathrm{tr}(x^tx) - 2)$. Writing $t = \mathrm{diag}(R_\theta, 1)$ with $R_\theta = \left(\begin{smallmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{smallmatrix}\right)$, we get that $\mathcal{O}_{at}(f_0)\det(1 - at|\mathfrak{n}_1)$ equals

$$\begin{aligned}2\int_{B^+} f_{0,a}(b^{-1}tb)(b^\alpha - b^{-\alpha})\,db &= 2\int_0^\infty f_{0,a}\left(\begin{pmatrix}e^{-s} & \\ & e^s\end{pmatrix} R_\theta \begin{pmatrix}e^s & \\ & e^{-s}\end{pmatrix}\right)(e^{2s} - e^{-2s})\,ds \\ &= 2\int_0^\infty f_{0,a}\begin{pmatrix}\cos\theta & e^{-2s}\sin\theta \\ e^{2s}\sin\theta & \cos\theta\end{pmatrix}(e^{2s} - e^{-2s})\,ds \\ &= 2\int_0^\infty \phi_{0,a}(2\cos^2 + (e^{4s} + e^{-4s})\sin^2 - 2)(e^{2s} - e^{-2s})\,ds \\ &= 2\int_0^\infty \phi_{0,a}(2\cos^2 + (e^{2s} + e^{-2s})^2\sin^2 - 2\sin^2 - 2)(e^{2s} - e^{-2s})\,ds \\ &= 2\int_0^\infty \phi_{0,a}(4\cos^2 + (e^{2s} + e^{-2s})^2\sin^2 - 4)(e^{2s} - e^{-2s})\,ds \\ &= \int_0^\infty \phi_{0,a}(4\cos^2 + (e^s + e^{-s})^2\sin^2 - 4)(e^s - e^{-s})\,ds\end{aligned}$$

$$
\begin{aligned}
&= \int_2^\infty \phi_{0,a}(4\cos^2\theta + v^2\sin^2\theta - 4)\,dv \\
&= \int_2^\infty \phi_{0,a}((v^2-4)\sin^2\theta)\,dv = \int_0^\infty \frac{\phi_{0,a}(y)}{2\sqrt{4\sin^2\theta + y}\,|\sin\theta|}.
\end{aligned}
$$

This extends to an m-times differentiable function in the point $\theta = 0$ if $\phi_{0,a}(y)$ vanishes to order m at $y = 0$. Let I denote the integral transform $I(h)(b) = \int_N h(bn)$, where h is a function on $M_1^0 \cong \mathrm{SL}_2(\mathbb{R})$, $b = \mathrm{diag}(e^s, e^{-s})$ and N is the group of all upper triangular matrices in $\mathrm{SL}_2(\mathbb{R})$ with ones on the diagonal. Then

$$
I(f_{0,a})(b) = \int_N \int_{N_1} f_0(bnan')\,dn\,dn' = \int_N \int_{N_1} f_0(abnn')\,dn\,dn' = \int_{N_{\mathrm{sp}}} f_0(abn)\,dn = g(ab)(ab)^{-\rho}.
$$

In terms of $\phi_{0,a}$, we have

$$
\begin{aligned}
I(f_{0,a})(b) = I(f_{0,a})\begin{pmatrix} e^s & \\ & e^{-s} \end{pmatrix} &= \int_{\mathbb{R}} \phi_{0,a}\left(\mathrm{tr}\left(\begin{pmatrix} e^s & \\ & e^{-s} \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) - 2\right)dx \\
&= \int_{\mathbb{R}} \phi_{0,a}(e^{2s} + e^{-2s} + e^{2s}x^2 - 2)\,dx = e^{-s}\mathcal{A}(\phi_{0,a})((e^s - e^{-s})^2),
\end{aligned}
$$

where $\mathcal{A}$ denotes the *Abel transform*

$$
\mathcal{A}(\phi)(y) = \int_{\mathbb{R}} \phi(y + x^2)\,dx.
$$

By [9, Lemma 11.2.7], the Abel transform is invertible; more precisely, one has the following: Let ϕ be a continuously differentiable function on $[0,\infty)$ such that $|\phi(x+s^2)|, |s\phi'(x+s^2)| \le \alpha(s)$ for some $\alpha \in L^1([0,\infty))$ and all $x \ge 0$. Then $q = \mathcal{A}(\phi)$ is continuously differentiable, and $\phi = \frac{-1}{\pi}\mathcal{A}(q')$. This implies that vanishing up to some order of the function g is inherited by the function $\phi_{0,a}$, and the lemma is proven. □

Pseudo-cusp forms

Definition 2.3. A K-finite function $f \in L^1(G)$ is called a *trace class function*, if $\pi(f)$ is a trace class operator for every $\pi \in \widehat{G}$. A trace class function is called a *pseudo-cusp form* if $\mathrm{tr}\,\pi(f) = 0$ for every $\pi \in \widehat{G}$, which is induced from the minimal parabolic P_{sp}. If π, η are representations of a group G and H is a subgroup, we write $[\pi : \eta]_H$ for $\dim \mathrm{Hom}_H(V_\pi, V_\eta)$.

Definition 2.4. For $N \in \mathbb{N}$, let $\mathcal{C}_N^{\mathrm{ev}}(\mathbb{R})$ denote the space of all even $\phi \in C^N(\mathbb{R})$ such that $\phi^{(m)}(x) = O(|x|^{-N})$ for every $0 \le m \le N$. Let $\phi \in \mathcal{C}_N^{\mathrm{ev}}(\mathbb{R})$. Let $\tau \in \widehat{K}$ be given. Then $\sup_{\pi \in \widehat{G}, V_\pi(\tau) \ne 0} \pi(C) < \infty$, where C is the Casimir operator and, as usual, we write $\pi(C)$ for the scalar that C acts through on the representation space V_π of π. By [10, Proposition 2.1], it follows that, for sufficiently large $u \in \mathbb{R}$, there exists a uniquely determined $f_{\phi,\tau,u} \in \mathcal{C}_N^1(G)$ such that

$$
\pi(f_{\phi,\tau,u}) = \frac{1}{\dim \tau}\phi\left(\sqrt{-\pi(C) - \frac{1}{4} + u}\right)P_{\pi,\tau}
$$

holds for every $\pi \in \widehat{G}$. The dimension factor is there to give this operator the trace $\phi(\sqrt{-\pi(C) - \frac{1}{4} + u})[\pi : \tau]_K$, where $[\pi : \tau]_K = \dim \mathrm{Hom}_K(V_\pi, V_\tau)$.

Let ρ_1 denote the half sum of positive roots of (A_1, P_1). Recall that, for every $k = 0, 1, 2, \ldots$, there exists an irreducible representation δ_{2k} of $K = \mathrm{SO}(3)$ of dimension $2k+1$, and this exhausts $\widehat{K}$. For a virtual K-representation $\tau = \tau_+ - \tau_-$, we define $f_{\phi,\tau,u} = f_{\phi,\tau_+,u} - f_{\phi,\tau_-,u}$. For every $n \in \mathbb{N}$, there is a discrete series representation $\mathcal{D}_{n+1}$ of $M_1 = \mathrm{SL}_2^{\pm}(\mathbb{R})$ which fits into an exact sequence

$$
0 \to \mathcal{D}_{n+1} \to \pi_{(-1)^{n+1},n} \to \sigma_{n-1} \to 0,
$$

where σ_m is the representation of $\mathrm{SL}_2^{\pm}(\mathbb{R})$ on the space of homogeneous polynomials $p(X, Y)$ of degree m. The $\mathcal{D}_{n+1}$ exhaust the discrete series of M_1. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to $\mathfrak{k} = \mathrm{Lie}(K)$.

For $k = 0, 1, 2, \ldots$, we consider the virtual representation

$$\tau_0 = \delta_4 - \delta_2 - 2\delta_0,$$
$$\tau_k = \tau_0 \otimes (\delta_{2k} - \delta_{2(k-1)} + \delta_{2(k-2)} - \cdots \pm \delta_0), \quad k \geq 1.$$

For simplicity, we write $f_{\phi,k,u}$ instead of $f_{\phi,\tau_k,u}$.

Lemma 2.5. *Let $\pi = \pi_{\mathcal{D}_{n+1},\nu}$ be induced from the discrete series representation $\mathcal{D}_{n+1}$ and $\nu \in \mathfrak{a}_{1,\mathbb{C}}^*$.*
(i) *Write $\mathfrak{p}_{M_1} = \mathfrak{p} \cap \mathfrak{m}_1$. Then $\mathrm{tr}\,\pi(f_{\phi,k,u})$ equals*

$$\mathrm{tr}\,\pi_{\mathcal{D}_{n+1},\nu}(f_{\phi,k,u}) = \begin{cases} \phi(\sqrt{|\nu|^2 + u}), & n = k+1, \\ 0 & \textit{otherwise}. \end{cases}$$

(ii) *If, on the other hand, $\pi \in \widehat{G}$ is induced from P_{sp}, then $\mathrm{tr}\,\pi(f_{\phi,k,u}) = 0$, so $f_{\phi,k,u}$ is a pseudo-cusp form.*

Proof. For $t \in T_1 \cong \mathrm{SO}(2)$ of the form $t = \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$, we write $\varepsilon_m(t) = (a + bi)^m$. Lemma 5.1 in [10] tells us that, in the case $k = 0$, we have

$$\mathrm{tr}\,\pi_{\mathcal{D}_{n+1},\nu}(f_{\phi,0,u}) = \phi(\sqrt{|\nu|^2 + u})\Big[\mathcal{D}_{n+1}, \Big(\bigwedge^{\mathrm{odd}} \mathfrak{p}_{M_1} - \bigwedge^{\mathrm{ev}} \mathfrak{p}_{M_1}\Big)\Big]_{K \cap M_1^+}.$$

An inspection of the proof yields that the expression $\mathrm{tr}\,\pi_{\mathcal{D}_{n+1},\nu}(f_{\phi,\tau_0 \otimes \delta_{2k},u})$ equals $\phi(\sqrt{|\nu|^2 + a})$ times

$$\Big[\mathcal{D}_{n+1}, \Big(\bigwedge^{\mathrm{odd}} \mathfrak{p}_{M_1} - \bigwedge^{\mathrm{ev}} \mathfrak{p}_{M_1}\Big) \otimes \delta_{2k}\Big]_{K \cap M_1^+}.$$

The restriction to $\mathrm{SO}(2)$ equals

$$\begin{aligned}\Big(\bigwedge^{\mathrm{odd}} \mathfrak{p}_{M_1} - \bigwedge^{\mathrm{ev}} \mathfrak{p}_{M_1}\Big) \otimes \delta_{2k}\Big|_{\mathrm{SO}(2)} &= (\varepsilon_2 + \varepsilon_{-2} - 2\varepsilon_0)(\varepsilon_k + \varepsilon_{k-1} + \cdots + \varepsilon_{-k}) \\ &= (\varepsilon_{k+2} + \varepsilon_{-(k+2)}) + (\varepsilon_{k+1} + \varepsilon_{-(k+1)}) - (\varepsilon_k + \varepsilon_{-k}) - (\varepsilon_{k-1} + \varepsilon_{-(k-1)}).\end{aligned}$$

We add these contributions up with alternate signs to get

$$\mathrm{tr}\,\pi_{\mathcal{D}_{n+1},\nu}(f_{\phi,k,u}) = \phi(\sqrt{|\nu|^2 + u})[\mathcal{D}_{n+1} : (\varepsilon_{k+2} + \varepsilon_{-(k+2)}) - (\varepsilon_k - \varepsilon_{-k})]_{K \cap M_1}.$$

Now, since

$$\mathcal{D}_{n+1}|_{\mathrm{SO}(2)} = \bigoplus_{\substack{|k| \geq n+1 \\ k \equiv (n+1) \bmod 2}} \varepsilon_k,$$

we find that there is only one n with a non-zero contribution, and this is $n = k + 1$. This implies the first claim. The second is obtained as in [10, Lemma 5.1]. □

We next compute orbital integrals for $f_{\phi,k,u}$.

Lemma 2.6. *Let $\phi \in \mathcal{C}_N^{\mathrm{ev}}(\mathbb{R})$ and $a \in \mathbb{R}$ be sufficiently large. The function $F_{f_{\phi,k,u}}^{A_{\mathrm{sp}} T_{\mathrm{sp}}}$ vanishes identically. For $at \in A_1 T_1$, we have $\Delta_I = (1 - \varepsilon_{-2})$ and, for $k = 0, 1, 2, \ldots$, we have*

$$F_{f_{\phi,k,u}}^{A_1 T_1}(at) = \Delta_I \frac{\varepsilon_{k+1}(t) + \varepsilon_{-k-1}(t)}{2} \int_{\mathfrak{a}_1^*} \phi(\sqrt{|\nu|^2 + u}) a^{i\nu}\, d\nu.$$

Proof. The first assertion is due to the fact that $f_{\phi,k,u}$ is a pseudo-cusp form. It follows that

$$F_{f_{\phi,k,u}}^{A_1 T_1}(at) = \frac{1}{2} \sum_{\chi \in \widehat{T}_1} \int_{\mathfrak{a}_1^*} \big(\langle (\chi, i\nu), at \rangle - \langle (\overline{\chi}, i\nu), at \rangle \varepsilon_{-2}(t)\big)\, \mathrm{tr}\,\pi_{\overline{\chi},\nu}(f_{\phi,k,u})\, d\nu.$$

Here $\tilde{\chi}$ stands for the (limit of) discrete series representation with Harish-Chandra parameter χ. For $\chi = \varepsilon_0$, this is the Steinberg representation, on which $f_{\phi,k}$ has trace zero. For $\chi = \varepsilon_{\pm m}$ with $m \geq 1$, we have $\tilde{\chi} = \mathcal{D}_{m+1}$. By the last lemma, only the summands with $|m| = k + 1$ give non-zero contributions. So the sum is

$$\int_{\mathfrak{a}_1^*} \phi(\sqrt{|\nu|^2 + u})a^{i\nu}\, d\nu$$

times $\varepsilon_{k+1} - \varepsilon_{-k-3} + \varepsilon_{-k-1} - \varepsilon_{k-1} = (1 - \varepsilon_{-2})(\varepsilon_{k+1} + \varepsilon_{-k-1})$. □

3 The test function

Symmetrized orbital integrals

Let $f_0 \in \mathcal{C}_N^1(K\backslash G/K)$ for N large enough. The function $F_{f_0}^{A_1T_1}(at)\Delta_I(t)^{-1}$ is even in $t \in T_1$, and therefore it has a Fourier expansion

$$\frac{F_{f_0}^{A_1T_1}(at)}{\Delta_I(t)} = \tilde{C}(a) + \sum_{k=0}^{\infty}(\varepsilon_{k+1}(t) + \varepsilon_{-k-1}(t))\tilde{C}_k(a),$$

which is such that $\tilde{C}, \tilde{C}_k$ become arbitrarily smooth as N increases and such that, for every $m \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that

$$|\tilde{C}_k(a)| \ll k^{-m}$$

uniformly. The function Δ_+ is symmetric in a in the sense that $\Delta_+(at) = \Delta_+(a^{-1}t)$ for $at \in A_1T_1$. We make the invariant integral artificially symmetric by considering $SF_{f_0}^{A_1T_1}(at) = F_{f_0}^{A_1T_1}(at) + F_{f_0}^{A_1T_1}(a^{-1}t)$. We then have

$$\frac{SF_{f_0}^{A_1T_1}(at)}{\Delta_I(t)} = C(a) + \sum_{k=0}^{\infty}(\varepsilon_{k+1}(t) + \varepsilon_{-k-1}(t))C_k(a),$$

where C, C_k are now symmetric, too. For every $k \geq 0$, there exists $u_k \in \mathbb{R}$ and an even function $\phi_k \in \mathcal{C}_N^{\mathrm{ev}}(R)$ such that $C_k(a) = \int_{\mathfrak{a}_1^*} \phi_k(\sqrt{|\nu|^2 + u_k})a^{i\nu}\, d\nu$. Then the function f_{ϕ_k,k,u_k} does not depend on the choice of the pair (ϕ_k, u_k). We therefore write it as $f_{[k]}$. Let

$$f_1 = -\sum_{k=0}^{\infty} f_{[k]}.$$

The sum converges uniformly with all derivatives up to order N.

Let $S\mathcal{O}_\gamma(h) = \mathcal{O}_\gamma(h) + \mathcal{O}_{\gamma^{-1}}(h)$ denote the symmetrized orbital integral of a function h.

Lemma 3.1. *Suppose that $g \in \mathcal{C}_N(A_{\mathrm{sp}})^W$ vanishes on the walls of the Weyl chambers up to order one. Let $f_0 = \Phi^{-1}(g) \in \mathcal{C}_N^1(K\backslash G/K)$, where Φ is the map of Proposition 1.3. Let f_1 be as above, and set $f = f_0 + f_1 \in \mathcal{C}_N^1(G)$. Then, for any semisimple element $\gamma \in G$, we have $S\mathcal{O}_\gamma(f) = 0$ unless γ is conjugate to an element $at \in A_{\mathrm{sp}}^+T_{\mathrm{sp}}$, where A_{sp}^+ is the open positive Weyl chamber, in which case we call γ a* split regular element, *and then we have*

$$S\mathcal{O}_\gamma(f) = \frac{g(a) + g(a^{-1})}{a^\rho \det(1 - (at)^{-1}|\mathfrak{n}_{\mathrm{sp}})}.$$

Proof. We have $F_f^{A_{\mathrm{sp}}T_{\mathrm{sp}}} = F_{f_0}^{A_{\mathrm{sp}}T_{\mathrm{sp}}}$ since f_1 is a pseudo-cusp form. By Lemma 2.6, it follows that

$$\frac{SF_f^{A_1T_1}(at)}{\Delta_I(t)} = C(a),$$

for regular elements at of A_1T_1, i.e., this symmetric invariant integral is independent of t. Let $t(\theta) \in T_1$, where θ is the angle of the rotation. Then $\Delta_I(at(\theta)) = (1 - e^{-2i\theta})$. Let $\omega = \omega_a$, where $a \in A_1$ is regular. Then, by Lemma 1.7, we have

$$S\mathcal{O}_a(f) = \frac{\omega(C(a)\Delta_I(t(\theta)))|_{\theta=0}}{a^{\rho_1}\det(1 - a^{-1}|(\mathfrak{g}/\mathfrak{g}_a)^+)} = \frac{2C(a)}{a^{\rho_1}\det(1 - a^{-1}|\mathfrak{n}_1)},$$

where $P_1 = A_1 M_1 N_1$ is the parabolic which contains P_{sp} and has A_1 as a split component. On the other hand, if $b = b_t = \operatorname{diag}(e^t, e^{-t}, 1)$ with $t \neq 0$, then ab_t is a regular element of A_{sp}, and then $\mathcal{O}_{ab}(f) = \mathcal{O}_{ab}(f_0)$ as f_1 is a pseudo-cusp form. Therefore, again by Lemma 1.7, we get

$$\mathcal{O}_a(f) = \frac{\frac{d}{dt}\big|_{t=0} f_0^{N_{sp}}(ab_t)}{a^{\rho_1}\det(1 - a|\mathfrak{n}_1)}.$$

It follows

$$C(a) = \frac{1}{2}\frac{d}{dt}\bigg|_{t=0} f_0^{N_{sp}}(ab_t) + f_0^{N_{sp}}(a^{-1}b_t).$$

This vanishes since g vanishes on a up to order 1. The claim follows. □

The twisting character

The representation ring $\operatorname{Rep}(G)$ is freely generated by two representations:

- $\mathrm{st}\colon G \to GL_3(\mathbb{C})$, the standard representation,
- $\Lambda = \bigwedge^2 \mathrm{st}$, its exterior square.

Let $\eta = (\Lambda + \mathrm{st} + 2) \otimes (\Lambda - \mathrm{st})$ as an element of $\operatorname{Rep}(G)$, i.e., a virtual representation. Recall that the trace map $\pi \mapsto \operatorname{tr}\pi$ is a ring homomorphism from $\operatorname{Rep}(G)$ to $C(G)$, the algebra of continuous functions on G.

Lemma 3.2. *For given $\pi \in \operatorname{Rep}(G)$, the trace $\pi(x)$ is a symmetric polynomial in the eigenvalues u, v, w of x. This constitutes a ring isomorphism*

$$\operatorname{Rep}(G) \cong \mathbb{Z}[u, v, w]^{\operatorname{Per}(3)}/(uvw - 1).$$

If u, v, w are the complex eigenvalues of $x \in G$, then $uvw = 1$ and $\operatorname{tr}\eta(x) = (u^2 - 1)(v^2 - 1)(w^2 - 1)$. We have $\operatorname{tr}\eta(x^{-1}) = -\operatorname{tr}\eta(x)$.

Proof. The first assertion is standard; the second is a matter of a computation involving st and Λ given by the symmetric polynomials $u + v + w$ and $uv + uw + vw$. The equation $\operatorname{tr}\eta(x) = -\operatorname{tr}\eta(x^{-1})$ uses $uvw = 1$ and follows from $\eta(x) = uvw(u - u^{-1})(v - v^{-1})(w - w^{-1})$. □

The geometric side

Let α, β denote the simple roots of A_{sp} given by

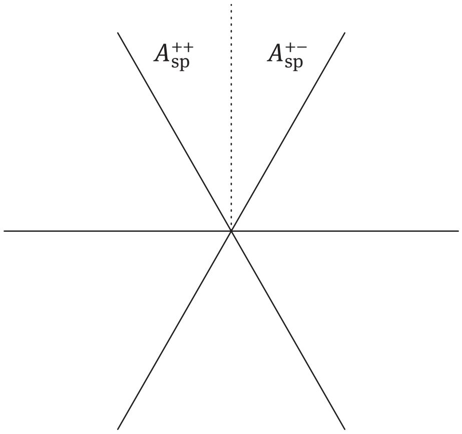
$$\operatorname{diag}(u, v, w)^\alpha = \frac{u}{v}, \quad \operatorname{diag}(u, v, w)^\beta = \frac{v}{w}.$$

We consider the set A_{sp}^{++} of all $a \in A_{sp}^+$ such that $a^\alpha > a^\beta$. In coordinates, the set A_{sp}^+ is the set of diagonal matrices $\operatorname{diag}(u, v, w)$ with $u, v, w > 0$ satisfying $uvw = 1$ and $u > v > w$. The set A_{sp}^{++} is the subset of all $\operatorname{diag}(u, v, w)$ for which additionally $v < 1$. As A_{sp}^+ is a fundamental domain in A_{sp} for the action of the Weyl group, the set A_{sp}^{++} is a fundamental domain for the action of the group generated by the Weyl group and the transformation $a \mapsto a^{-1}$. Likewise, let A_{sp}^{+-} denote the set of all $a \in A_{sp}^+$ with $a^\alpha < a^\beta$.

Proposition 3.3. *Let $g \in \mathcal{C}_N(A_{sp})^W$, and suppose that g vanishes on A_{sp}^{+-}. Let f be as in Lemma 3.1. Let Γ be a congruence subgroup of $SL_3(\mathbb{Q})$, and let K_Γ be the closure of Γ in $SL_3(\mathbb{A}_{fin})$, where $\mathbb{A}_{fin}$ is the ring of finite $\mathbb{Q}$-adeles. Let f_{fin} be the characteristic function of K_Γ, and let $f_\mathbb{A} = f_{fin} \otimes f \operatorname{tr}\eta$ as a function on the adeles, where η is the twisting character. Then the geometric side of the Arthur trace formula of f equals*

$$J_{geom}(f_\mathbb{A}) = \sum_{[\gamma]} \operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \frac{g(a_\gamma)\operatorname{tr}\eta(\gamma)}{a_\gamma^\rho \det(1 - (a_\gamma t_\gamma)^{-1}|\mathfrak{n}_{sp})},$$

where the sum runs over all conjugacy classes $[\gamma]$ in Γ of split regular elements.



Proof. This summarizes the above computations and [10, Corollary 1.3], except for one point. We have only computed the symmetrized orbital integrals instead of orbital integrals. So $J_{\mathrm{geom}}(f_{\mathbb{A}})$ equals

$$\sum_{[\gamma]} \mathrm{vol}(\Gamma_\gamma \backslash G_\gamma) \mathcal{O}_\gamma(f) \operatorname{tr} \eta(\gamma).$$

As the orbital integrals vanish otherwise, we only have to consider those $\gamma \in \Gamma$ which are G-conjugate to some $a_\gamma t_\gamma \in A_{sp}^{++} M_{sp}$. But then

$$S\mathcal{O}_\gamma(f) = \frac{g(a_\gamma) \operatorname{tr} \eta(\gamma) + g(a_{\gamma^{-1}}) \operatorname{tr} \eta(\gamma^{-1})}{a_\gamma^\rho \det(1 - (a_\gamma t_\gamma)^{-1} | \mathfrak{n}_{sp})} = \frac{g(a_\gamma) \operatorname{tr} \eta(\gamma)}{a_\gamma^\rho \det(1 - (a_\gamma t_\gamma)^{-1} | \mathfrak{n}_{sp})}$$

since $g(a_{\gamma^{-1}}) = 0$. □

4 Prime geodesic theorem

A test function

For $s \in \mathbb{C}^2$ and $a \in A_{sp}^{++}$, write

$$a^{-s} = a^{-s_1(\alpha - \beta) - 2 s_2 \beta}.$$

This normalization has been chosen as $\alpha - \beta$ and 2β are the dual basis of the canonical basis of $\mathfrak{a}_{sp}^{++}$, which is the inverse image of A_{sp}^{++} under the exponential map. For given $j \in \mathbb{N}$ and $s \in \mathbb{C}^2$, let $g_{j,s} \in C(A_{sp})^W$ be defined by

$$g_{j,s}(a) = \begin{cases} \left(\frac{\partial}{\partial s_1} \frac{\partial}{\partial s_2}\right)^{j+1} a^{-s}, & a \in A_{sp}^{++}, \\ 0, & a \in A_{sp}^{+-}. \end{cases}$$

Lemma 4.1. *Let $f_{j,s} = f_0 + f_1$ be the function attached to $g = g_{j,s}$ as in Lemma 3.1. For every $N \in \mathbb{N}$, there exists $C > 0$ such that, for every $s \in \mathbb{C}^2$ with $\mathrm{Re}(s_1), \mathrm{Re}(s_2) > C$ and every $j \geq N$, one has $f_{j,s} \in \mathcal{C}_N^1(G)$.*

Proof. This follows from Proposition 1.6 (b). □

We now specify the multiple B of the Killing form which fixes the Haar measures. We choose this in a way that the map $A \to \mathbb{R}^2$, $a \mapsto ((\alpha - \beta)(\log a), \beta(\log a))$ is measure-preserving.

Lemma 4.2. *Let π_{triv} denote the trivial representation of G. Write D for the differential operator $\frac{\partial}{\partial s_1} \frac{\partial}{\partial s_2}$. We have*

$$\operatorname{tr}(\pi_{\mathrm{triv}}(f_{j,s} \operatorname{tr} \eta)) = \frac{1}{2} D^{j+1} \left(\frac{1}{(s_1 - \frac{7}{3})(s_2 - 2)} - \frac{1}{(s_1 - \frac{5}{3})(s_2 - 2)} \right) + F(s),$$

where F is holomorphic in $\{\mathrm{Re}(s_1) > \frac{7}{3} - \varepsilon, \mathrm{Re}(s_2) > 2 - \varepsilon\}$ for some $\varepsilon > 0$.

Proof. Note first that

$$\operatorname{tr}\eta(a) = (a^{\frac{4}{3}(\alpha-\beta)+2\beta} - 1)(a^{-\frac{2}{3}(\alpha-\beta)} - 1)(a^{-\frac{2}{3}(\alpha-\beta)-2\beta} - 1).$$

As $a^{\alpha-\beta}$ and a^{β} both tend to infinity, the leading term is $a^{\frac{4}{3}(\alpha-\beta)+2\beta}$, followed by a summand $-a^{\frac{2}{3}(\alpha-\beta)+2\beta}$. All further summands have lower growth rate in both, $\alpha-\beta$ and β. Using Weyl's integration formula, we compute

$$\begin{aligned}\operatorname{tr}\pi_{\mathrm{triv}}(f_{j,s}\operatorname{tr}\eta) &= \int_G f_{j,s}(x)\operatorname{tr}\eta(x)\,dx\\ &= \int_{A_{\mathrm{sp}}^{++}} \frac{1}{4}\sum_{t\in T_{\mathrm{sp}}} \mathcal{O}_{at}(f_{j,s})\operatorname{tr}\eta(at)|D_{\mathrm{sp}}(at)|^2\,da + \frac{1}{2}\int_{H_1} \underbrace{\mathcal{O}_h(f_{j,s})}_{=0}\operatorname{tr}\eta(h)|D_1(h)|^2\,dh.\end{aligned}$$

By our computation of the orbital integrals, this equals

$$\int_{A_{\mathrm{sp}}^{++}} g_{j,s}(a)\frac{1}{4}\sum_{t\in T_{\mathrm{sp}}} a^{\rho}\det(1-(at)^{-1}|\mathfrak{n}_{\mathrm{sp}})\operatorname{tr}\eta(a)\,da = \int_{A_{\mathrm{sp}}^{++}} g_{j,s}(a)(a^{\alpha+\beta} - a^{-\alpha-\beta})\operatorname{tr}\eta(a)\,da.$$

So the leading term in $\operatorname{tr}\pi_{\mathrm{triv}}(f_{j,s}\operatorname{tr}\eta)$ is

$$\begin{aligned}D^{j+1}\int_{A_{\mathrm{sp}}^{++}} a^{-s_1(\alpha-\beta)-2s_2\beta}a^{\frac{7}{3}(\alpha-\beta)+4\beta}\,da &= D^{j+1}\int_0^\infty\int_0^\infty e^{(\frac{7}{3}-s_1)x+2(2-s_2)y}\,dx\,dy\\ &= \frac{1}{2}D^{j+1}\frac{1}{(s_1-\frac{7}{3})(s_2-2)}.\end{aligned}$$

Taking the next term in the asymptotic of $\operatorname{tr}\eta(a)$ into account gives the second term of the expansion in the lemma. □

Lemma 4.3. *Let J_{spec} denote the spectral side of the trace formula. There exists $\varepsilon > 0$ such that*

$$J_{\mathrm{spec}}(\mathbf{1}_{K_{\mathrm{fin}}}\otimes f_{j,s}\operatorname{tr}\eta) - \operatorname{tr}\pi_{\mathrm{triv}}(\mathbf{1}_{K_{\mathrm{fin}}}\otimes f_{j,s}\operatorname{tr}\eta)$$

converges locally uniformly absolutely for $s\in\Omega_\varepsilon$.

Proof. The spectral side $J_{\mathrm{spec}}(f)$ is a sum $\sum_\chi J_\chi(f)$, where χ runs through the set of conjugacy classes of pairs $(\mathcal{M}_0,\pi_0)$ consisting of a $\mathbb{Q}$-rational Levi subgroup $\mathcal{M}$ and a cuspidal representation π_0 of $\mathcal{M}_0$, the sum being absolutely convergent. The particular terms have expansions

$$J_\chi(f) = \sum_{\mathcal{M},\pi} J_{\chi,\mathcal{M},\pi}(f),$$

running over all $\mathbb{Q}$-rational Levi subgroups $\mathcal{M}$ containing a fixed minimal one and, for each $\mathcal{M}$, over all discrete automorphic representations π of $\mathcal{M}$. Explicitly,

$$J_{\chi,\mathcal{M},\pi}(f) = \sum_{w\in W_{\mathcal{M}}} c_{\mathcal{M},w}\int_{i\mathfrak{a}_{\mathcal{L}}^*}\sum_{\mathcal{P}}\operatorname{tr}(\mathfrak{M}_{\mathcal{L}}(\mathcal{P},\nu)M(\mathcal{P},w)\rho_{\chi,\pi}(\mathcal{P},\nu,f))\,d\nu.$$

Here, for a given element w of the Weyl group of $\mathcal{M}$, the Levi subgroup $\mathcal{L}$ is determined by $\mathfrak{a}_{\mathcal{L}} = (\mathfrak{a}_{\mathcal{M}})^w$, and $\mathcal{P}$ runs through all parabolic subgroups having $\mathcal{M}$ as Levi component. The coefficient $c_{\mathcal{M},w} > 0$ equals 1 in the case $\mathcal{M} = \mathcal{G} = SL_3$. We write $J^+_{\chi,\mathcal{M},\pi}(f)$ for the same expression with the trace replaced by the trace norm.

On G, our chosen multiple B of the Killing form induces a left-invariant metric. The corresponding Laplace–Beltrami operator $\Delta\geq 0$ is given by $\Delta = -C + 2C_K$, where C is the Casimir of G and C_K is the Casimir of K induced by the form B. Then the operator $(\Delta+1)^{-m}$ has a kernel which is $2m-\dim G$ times continuously differentiable. Further, the operator being left-invariant, it equals the convolution operator given by a function on G. In the proof of [15, Theorem 3], in particular formula (5.1), it is shown that $J^+_{\mathrm{spec}}(\mathbf{1}_{K_{\mathrm{fin}}}\otimes(\Delta+1)^{-m}) < \infty$ for m sufficiently large. Writing $f = (\Delta+1)^{-m}(\Delta+1)^m f$ and using the estimate $\|ST\|_{\mathrm{tr}}\leq\|S\|_{\mathrm{tr}}\|T\|_{\mathrm{op}}$, where S and T are operators and the norms are the trace norm and the operator norm, we see that it suffices to show that, for j and m sufficiently large, there exists an $\varepsilon > 0$ and a uniform bound on the operator norms of $\pi((\Delta+1)^m f_{j,s} \operatorname{tr}\eta)$ for all $s \in \Omega_\varepsilon$ and all non-trivial $\pi \in \widehat{G}$. Now

$$\begin{aligned}\pi((\Delta+1)^m f_{j,s} \operatorname{tr}\eta) &= \pi(\Delta+1)^m \pi(f_{j,s} \operatorname{tr}\eta)\\ &= \pi(\Delta+1)^m (1\otimes \operatorname{tr})((\pi\otimes\eta)(f_{j,s}))\\ &= (1\otimes\operatorname{tr})(\pi(\Delta+1)^m\otimes 1)((\pi\otimes\eta)(f_{j,s})).\end{aligned}$$

We further have $\pi(\Delta+1)^m = (\pi(\Delta)+1)^m = (-\pi(C)+2\pi(C_K)+1)^m$. Now $\pi(C)$ is a scalar, and $\pi(C_K)$ is scalar on each K-type. These scalars grow polynomially in k for the enumeration (δ_k) of K-types. But the sequence $f_{[k]}$ tends to zero faster than any power of k. This reduces the proof to a single K-type, where the proof uses the explicit knowledge on $\pi(f_{j,s})$ for $\pi \in \widehat{G}$ and proceeds analogously to the proof of [10, Lemma 8.3]. □

Definition 4.4. Let $\mathcal{E}(\Gamma)$ denote the set of all conjugacy classes $[\gamma]$ in Γ such that γ is in G conjugate to an element a_γ in A_{sp}^{++}. For $[\gamma] \in \mathcal{E}(\Gamma)$, the element γ defines a closed geodesic in $\Gamma\backslash X$, where $X = G/K$ is the symmetric space attached to G. This closed geodesic lies in a unique two-dimensional flat sub-manifold F_γ of $\Gamma\backslash X$. Let λ_γ be the metric volume of F_γ.

Theorem 4.5 (Prime geodesic theorem). *Let*

$$\Lambda(T_1,T_2) = \sum_{\substack{[\gamma]\in\mathcal{E}(\Gamma)\\ a_\gamma^{\alpha-\beta}\le T_1\\ a_\gamma^{2\beta}\le T_2}} \lambda_\gamma.$$

Then, as $T_1, T_2 \to \infty$ independently, we get $\Lambda(T_1,T_2) \sim T_1T_2$.

In the light of the prime geodesic theorem for cocompact groups in [6], the coordinate 2β appears naturally. In the present case, we reduced the fundamental domain from A_{sp}^{+} to A_{sp}^{++} by taking inversion into account; this explains the coordinate $\alpha-\beta$.

Proof. First define

$$\operatorname{ind}(\gamma) = \frac{\lambda_\gamma}{\det(1-(a_\gamma m_\gamma)^{-1}|\mathfrak{n})} > 0.$$

For $a \in A_{\mathrm{sp}}^{++}$, let $l(a) = 2(\alpha-\beta)(\log a)\cdot\beta(\log a)$. For $j \in \mathbb{N}$, consider the Dirichlet series

$$L^j(s) = \sum_{[\gamma]\in\mathcal{E}(\Gamma)} \operatorname{ind}(\gamma)\operatorname{tr}\eta(\gamma) l(a_\gamma)^{j+1} a_\gamma^{-s} a_\gamma^{-\frac{4}{3}(\alpha-\beta)-2\beta}.$$

We have shown

$$L^j(s) = D^{j+1}\Big(\frac{1}{(s_1-1)(s_2-1)} - \frac{1}{(s_1-\frac{1}{3})(s_2-1)}\Big) + R(s),$$

where $R(s)$ is holomorphic on $\{\mathrm{Re}(s_1), \mathrm{Re}(s_2) > 1-\varepsilon\}$ for some $\varepsilon > 0$. Now recall

$$\operatorname{tr}\eta(a) = (a^{\frac{4}{3}(\alpha-\beta)+2\beta}-1)(a^{-\frac{2}{3}(\alpha-\beta)}-1)(a^{-\frac{2}{3}(\alpha-\beta)-2\beta}-1).$$

This implies

$$\operatorname{tr}\eta(\gamma)a_\gamma^{-\frac{4}{3}(\alpha-\beta)-2\beta} \to 1 \quad \text{for} \quad a_\gamma^{\alpha-\beta}, a_\gamma^{2\beta} \to \infty \quad \text{independently}.$$

The higher-dimensional Wiener–Ikehara theorem, i.e., [6, Theorem 3.2], implies that with

$$\tilde{\Lambda}(T_1,T_2) = \sum_{\substack{[\gamma]\in\mathcal{E}(\Gamma)\\ a_\gamma^{\alpha-\beta}\le T_1\\ a_\gamma^{2\beta}\le T_2}} \operatorname{ind}(\gamma)\operatorname{tr}\eta(\gamma)a_\gamma^{-\frac{4}{3}(\alpha-\beta)-2\beta},$$

we get $\tilde{\Lambda}(T_1,T_2) \sim T_1T_2$ as $T_1, T_2 \to \infty$ independently. Now

$$\frac{\operatorname{ind}(\gamma)\operatorname{tr}\eta(\gamma)a_\gamma^{-\frac{4}{3}(\alpha-\beta)-2\beta}}{\lambda_\gamma}$$

tends to 1 as $a_\gamma^{\alpha-\beta}$ and $a_\gamma^{2\beta}$ tend to ∞. So the prime geodesic theorem follows by [6, Lemma 3.5]. □

Application to class numbers

In this section, we give an asymptotic formula for class numbers of orders in number fields. It is quite different from known results like Siegel's theorem [2, Theorem 6.2]. The asymptotic is in several variables and thus contains more information than a single variable one. In a sense, it states that the units of the orders are equally distributed in different directions if only one averages over sufficiently many orders.

Let $O_{\mathbb{R}}(3)$ denote the set of all orders $\mathcal{O}$ in totally real number fields F of degree 3. For such an order $\mathcal{O}$, let $h(\mathcal{O})$ be its class number, $R(\mathcal{O})$ its regulator.

For $\lambda \in \mathcal{O}^\times$, let ρ_1, ρ_2, ρ_3 denote the real embeddings of F ordered in a way that $|\rho_1(\lambda)| \geq |\rho_2(\lambda)| \geq |\rho_3(\lambda)|$. Let

$$\alpha_1(\lambda) = \frac{|\rho_1(\lambda)\rho_3(\lambda)|}{|\rho_2(\lambda)|^2}, \quad \alpha_2(\lambda) = \left(\frac{|\rho_2(\lambda)|}{|\rho_3(\lambda)|}\right)^2.$$

Note that these are the multiplicative versions of the dual cone basis $(\alpha - \beta, 2\beta)$ of $\mathfrak{a}_{sp}^{++}$.

Theorem 4.6. *For $T_1, T_2 > 0$, set*

$$\vartheta(T) = \sum_{\substack{\mathcal{O} \in O_{\mathbb{R}}(3),\, \lambda \in \mathcal{O}^\times/\pm 1 \\ 1 < \alpha_1(\lambda) \leq T_1 \\ 1 < \alpha_2(\lambda) \leq T_2}} R(\mathcal{O}) h(\mathcal{O}).$$

Then we have, as $T_1, T_2 \to \infty$,

$$\vartheta(T_1, T_2) \sim \frac{16}{\sqrt{3}} T_1 T_2.$$

Proof. For $\lambda \in \mathcal{E}(\Gamma)$, where $\Gamma = SL_3(\mathbb{Z})$, the centralizer F_γ in $M_3(\mathbb{Q})$ is a totally real number field of degree 3, and all such occur in this way. Next the centralizer $\mathcal{O}_\gamma$ of γ in $M_3(\mathbb{Z})$ is an order in F_γ and, up to isomorphy, all totally real cubic orders occur, each with order $\mathcal{O}$ multiplicity $h(\mathcal{O})$ (see [12]). The number $16/\sqrt{3}$ occurs as renormalization factor between our measure on A_{sp} and the measure used for the regulator. □

5 A conjectural Lefschetz formula

Spectral Lefschetz numbers

Let G denote a connected semisimple Lie group with finite center. Fix a maximal compact subgroup K with Cartan involution θ. So θ is an automorphism of G with $\theta^2 = \mathrm{Id}$, and K is the set of all $x \in G$ with $\theta(x) = x$.

Let $\mathfrak{g}_{\mathbb{R}}, \mathfrak{k}_{\mathbb{R}}$ denote the real Lie algebras of G and K, and let $\mathfrak{g}$ and $\mathfrak{k}$ denote their complexifications. This will be a general rule: for a Lie group H, we denote by $\mathfrak{h}_{\mathbb{R}}$ the Lie algebra of H and by $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} \otimes \mathbb{C}$ its complexification. Let $b\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be a positive multiple of the Killing form. On G, K and all parabolic subgroups as well as all Levi components, we install Haar measures given by the form b as in [18].

Let H be a non-compact Cartan subgroup of G. Modulo conjugation, we can assume that $H = AB$, where A is a connected, θ-stable split torus and B is a closed subgroup of K. Fix a parabolic P with split component A. Then P has Langlands decomposition $P = MAN$, and B is a Cartan subgroup of M. Note that an arbitrary parabolic subgroup $P' = M'A'N'$ of G occurs in this way if and only if the group M' has a compact Cartan subgroup. In this case, we say that P' is a *cuspidal parabolic*.

The choice of the parabolic P amounts to the same as a choice of a set of positive roots $\phi^+(\mathfrak{g}, \mathfrak{a})$ in the root system $\phi(\mathfrak{g}, \mathfrak{a})$. The Lie algebra $\mathfrak{n}$ of the unipotent radical N can be described as $\mathfrak{n} = \bigoplus_{\alpha \in \phi^+(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the *root space* attached to α, i.e., $\mathfrak{g}_\alpha$ is the space of all $X \in \mathfrak{g}$ such that $\mathrm{ad}(Y)X = \alpha(Y)X$ holds for every $Y \in \mathfrak{a}$. Define $\bar{\mathfrak{n}} = \bigoplus_{\alpha \in \phi^+(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_{-\alpha}$. This is the *opposite* Lie algebra. Let $\bar{\mathfrak{n}}_{\mathbb{R}} = \bar{\mathfrak{n}} \cap \mathfrak{g}_{\mathbb{R}}$ and $\bar{N} = \exp(\bar{\mathfrak{n}}_{\mathbb{R}})$. Then $\bar{P} = MA\bar{N}$ is the *opposite parabolic* to P.

Let $\mathfrak{a}^*$ denote the dual space of $\mathfrak{a}$. Since $A = \exp(\mathfrak{a}_{\mathbb{R}})$, every $\lambda \in \mathfrak{a}^*$ induces a continuous homomorphism from A to $\mathbb{C}^*$ written $a \mapsto a^\lambda$ and given by $(\exp(H))^\lambda = e^{\lambda(H)}$. Let $\rho_P \in \mathfrak{a}^*$ be the half of the sum of all positive roots, each weighted with its multiplicity. So $a^{2\rho_P} = \det(a|\mathfrak{n})$. Let $\mathfrak{a}_{\mathbb{R}}^- \subset \mathfrak{a}_{\mathbb{R}}$ be the negative Weyl chamber consisting of all $X \in \mathfrak{a}_{\mathbb{R}}$ such that $\alpha(X) < 0$ for every $\alpha \in \phi^+(\mathfrak{g}, \mathfrak{a})$. Let $A^- = \exp(\mathfrak{a}_{\mathbb{R}}^-)$ be the negative Weyl chamber in A. Further, let $\overline{A^-}$ be the closure of A^- in A. This is a manifold with corners. Let $K_M = M \cap K$. Then K_M is a maximal compact subgroup of M, and it contains B. Fix an irreducible unitary representation (σ, V_σ) of K_M. Then V_σ is finite dimensional. Let $\breve{\sigma}$ be the dual representation to σ.

Let $\widehat{G}$ denote the unitary dual of G, i.e., it is the set of all isomorphy classes of irreducible unitary representations of G. Let $\widehat{G}_{\mathrm{adm}} \supset \widehat{G}$ be the admissible dual, i.e., the set of classes of admissible irreducible representations under infinitesimal equivalence. Harish-Chandra proved that two unitary irreducible representations are unitarily equivalent if and only if they are infinitesimally equivalent. Therefore, $\widehat{G}$ can be considered a subset of $\widehat{G}_{\mathrm{adm}}$. For $\pi \in \widehat{G}_{\mathrm{adm}}$, let π_K denote the $(\mathfrak{g}, K)$-module of K-finite vectors in π, and let $\Lambda_\pi \in \mathfrak{h}^*$ be a representative of the infinitesimal character of π. Let $H^\bullet(\mathfrak{n}, \pi_K)$ be the Lie algebra cohomology with coefficients in π_K. By [20], for each $q \geq 0$, the $(\mathfrak{a} \oplus \mathfrak{m}, K_M)$-module $H^q(\mathfrak{n}, \pi_K)$ is admissible of finite length, i.e., a Harish-Chandra module.

Definition 5.1. For $\lambda \in \mathfrak{a}^*$ and an A-module W, let W^λ be the generalized λ-eigenspace, i.e., W^λ is the set of all $w \in W$ such that there is $n \in \mathbb{N}$ with $(a - a^\lambda)^n w = 0$ for every $a \in A$. Let $\mathfrak{m} = \mathfrak{k}_M \oplus \mathfrak{p}_M$ be the Cartan decomposition of the Lie algebra $\mathfrak{m}$ of M. For $\pi \in \widehat{G}$ and $\lambda \in \mathfrak{a}^*$, let $L^\sigma_\lambda(\pi)$ denote the *spectral Lefschetz number* given by

$$L^\sigma_\lambda(\pi) = \sum_{p,q \geq 0} (-1)^{p+q+\dim N} \dim\big(H^q(\mathfrak{n}, \pi_K)^\lambda \otimes \wedge^p \mathfrak{p}_M \otimes \breve{\sigma}\big)^{K_M}.$$

Definition 5.2. For a given smooth and compactly supported function $f \in C_c^\infty(G)$, we define its Fourier transform $\hat{f}\colon \widehat{G} \to \mathbb{C}$ by $\hat{f}(\pi) := \operatorname{tr} \pi(f)$.

Proposition 5.3. *The following statements hold:*

(a) *For every $\phi \in C_c^\infty(A^-)$, there exists $f_\phi \in C_c^\infty(G)$ such that, for every $\pi \in \widehat{G}$,*

$$\widehat{f_\phi}(\pi) = \sum_{\lambda \in \mathfrak{a}^*} L^\sigma_\lambda(\pi) \hat{\phi}(\lambda),$$

where $\hat{\phi}$ is the Fourier transform of ϕ, i.e., $\hat{\phi}(\lambda) = \int_A \phi(a) a^\lambda \, da$.

(b) *The sum in* (a) *is finite; more precisely, the Lefschetz number $L^\sigma_\lambda(\pi)$ is zero unless there is an element w of the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ such that $\lambda = (w\Lambda_\pi)|_{\mathfrak{a}} - \rho_P$.*

Proof. The proof of part (a) is contained in [7], and (b) is a consequence of [20, Corollary 3.32]. □

Geometric Lefschetz numbers

Let $\Gamma \subset G$ be a discrete subgroup of finite covolume, $X = G/K$ the symmetric space, and $X_\Gamma = \Gamma\backslash X = \Gamma\backslash G/K$ the corresponding locally symmetric quotient.

Fix a parabolic $P = MAN$ such that M has a compact Cartan B. Let H be the Cartan subgroup AB of G. Let A^- be the cone in A given by all elements a with $a^\alpha < 1$ for all roots α of (A, P). Let $\mathcal{E}_P(\Gamma)$ denote the set of all conjugacy classes $[\gamma]$ in Γ such that γ is in G conjugate to an element $a_\gamma b_\gamma$ of A^-B. Then there is a conjugate H_γ of H such that $\gamma \in H_\gamma$.

For $[\gamma] \in \mathcal{E}_P(\Gamma)$, we define the *geometric Lefschetz number* by

$$L^\sigma(\gamma) = \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma) \frac{\operatorname{tr} \sigma(b_\gamma)}{\det(1 - a_\gamma b_\gamma|\mathfrak{n})}.$$

The Lefschetz formula

The unitary G-representation on $L^2(\Gamma\backslash G)$ decomposes as $L^2(\Gamma\backslash G) = L^2_{\mathrm{disc}} \oplus L^2_{\mathrm{cont}}$, where

$$L^2_{\mathrm{disc}} = \bigoplus_{\pi \in \widehat{G}} N_\Gamma(\pi)\pi$$

is a direct sum of irreducibles with finite multiplicities and L^2_{cont} is a sum of continuous Hilbert integrals. In particular, L^2_{cont} does not contain any irreducible sub-representation.

Let r be the dimension of A, and let $\alpha_1, \dots, \alpha_r \in \mathfrak{a}_{\mathbb{R}}^*$ be the primitive positive roots. Let

$$\mathfrak{a}_{\mathbb{R}}^{*,+} = \{t_1\alpha_1 + \cdots + t_r\alpha_r : t_1, \dots, t_r > 0\}$$

be the positive dual cone, and let $\overline{\mathfrak{a}_{\mathbb{R}}^{*,+}}$ be its closure in $\mathfrak{a}_{\mathbb{R}}^*$.

For $\mu \in \mathfrak{a}^*$ and $j \in \mathbb{N}$, let $C^{\mu,j}(A^-)$ denote the space of all functions on A which

- are j-times continuously differentiable on A,
- are zero outside A^-,
- satisfy $|D\phi| \le C|a^\mu|$ for every invariant differential operator D on A of degree $\le j$, where $C > 0$ is a constant, which depends on D.

This space can be topologized with the seminorms

$$N_D(\phi) = \sup_{a \in A} |a^{-\mu} D\phi(a)|,$$

where $D \in U(\mathfrak{a})$ and $\deg(D) \le j$. Since the space of operators D as above is finite dimensional, one can choose a basis $D_1, \dots, D_n$ and set $\|\phi\| = N_{D_1}(\phi) + \cdots + N_{D_n}(\phi)$. The topology of $C^{\mu,j}(A^-)$ is given by this norm, and thus $C^{\mu,j}(A^-)$ is a Banach space.

Conjecture 5.4 (Lefschetz formula). *For $\lambda \in \mathfrak{a}^*$ and $\pi \in \widehat{G}_{\mathrm{adm}}$, there is an integer $N_{\Gamma,\mathrm{cont}}(\pi, \sigma, \lambda)$ which vanishes if $\mathrm{Re}(\lambda) \notin \overline{\mathfrak{a}_{\mathbb{R}}^{*,+}}$ and there are $\mu \in \mathfrak{a}^*$ and $j \in \mathbb{N}$ such that, for each $\phi \in C^{\mu,j}(A^-)$ and with*

$$\tilde{N}_\Gamma(\pi, \sigma, \lambda) = N_\Gamma(\pi) + N_{\Gamma,\mathrm{cont}}(\pi, \sigma, \lambda),$$

we have

$$\sum_{\substack{\pi \in \widehat{G} \\ \lambda \in \mathfrak{a}^*}} \tilde{N}_\Gamma(\pi, \sigma, \lambda) L_\lambda^\sigma(\pi) \int_A \phi(a) a^\lambda \, da = \sum_{[\gamma] \in \mathcal{E}_P(\Gamma)} L^\sigma(\gamma)\phi(a_\gamma).$$

Either side of this identity represents a continuous functional on $C^{\mu,j}(A^-)$.

If this conjecture was known, plugging in the test function of Section 4 would yield the Dirichlet series of the proof of Theorem 4.5 on the right-hand side. The left-hand side would then express this function $L(s)$ as a Mittag–Leffler series similar to [6, Theorem 2.1], which would yield analytic continuation.

In the following cases, the conjecture is known.

(i) The conjecture holds if Γ is cocompact. In that case, the numbers $N_{\Gamma,\mathrm{cont}}(\pi, \sigma, \lambda)$ are all zero. This is shown in [7].

(ii) In the next section, we will prove the conjecture for $G = \mathrm{PGL}_2(\mathbb{R})$.

6 PGL(2)

Let $G = \mathrm{PGL}_2(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})/\mathbb{R}^\times = \mathrm{SL}_2^\pm(\mathbb{R})/\pm 1$ and, for convenience, write $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ for the element $\mathbb{R}^\times \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of G. We further write $K = \mathrm{PO}(2) = \mathrm{O}(2)/\pm 1$ as well as

$$A = \left\{ \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix} : t \in \mathbb{R} \right\}, \quad N = \left\{ \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} : x \in \mathbb{R} \right\}.$$

Then $P = AN$ is a minimal parabolic subgroup of G. We write $\mathfrak{a}_{\mathbb{R}}$ for the real Lie algebra of A and $\mathfrak{a}$ for its complexification. The dual space $\mathfrak{a}^*$ can be identified with the set of continuous group homomorphisms $A \to \mathbb{C}^*$; we write $a \mapsto a^\phi$ for the element given by $\phi \in \mathfrak{a}^*$. Let $\rho \in \mathfrak{a}^*$ be the modular shift, i.e., $\rho\left(\begin{smallmatrix} t & \\ & -t \end{smallmatrix}\right) = t$. We identify $\mathfrak{a}^*$ with $\mathbb{C}$ by sending ρ to $\frac{1}{2}$. So any $\phi \in \mathfrak{a}^*$ will be written as $\phi = 2\lambda\rho$ with $\lambda \in \mathbb{C}$.

Let $\widehat{G}_{\mathrm{adm}}$ denote the set of all irreducible admissible continuous Banach representations of G up to infinitesimal equivalence, or, what amounts to the same, the set of all irreducible admissible $(\mathfrak{g}, K)$-modules up to isomorphy, where $\mathfrak{g}$ is the complexified Lie algebra of G (see [26]). We will describe the set $\widehat{G}_{\mathrm{adm}}$ below. First we give some representations.

- The finite-dimensional representations are δ_{2k} and $\det \otimes \delta_{2k}$ for an integer $k \geq 0$, where det stands for the determinant on $SL_2^{\pm}(\mathbb{R})$, and δ_{2k}, $k \in \mathbb{N}_0$, is the $2k+1$-dimensional representation on the space of G on the space of all polynomials $p(X,Y)$ in two variables which are homogeneous of degree $2k$. A basis of this space is $X^{2k}, X^{2k-1}Y, \dots, Y^{2k}$. The representation is $\delta_{2k}(g)p(X,Y) = p((X,Y)g)$.
- Let M be the subgroup of G consisting of two elements, the non-trivial being $T = \left[\begin{smallmatrix}-1 & \\ & 1\end{smallmatrix}\right]$. Let $\sigma \in \widehat{M}$ be a character. For $\lambda \in \mathbb{C}$, let $\pi_{\sigma,2\lambda\rho}$ be the representation on the space of all functions $f\colon G \to \mathbb{C}$ with $f(manx) = \sigma(m)a^{2\lambda\rho+\rho}f(x)$ such that $f|_K \in L^2(K)$. The representation is given by $\pi_{\sigma,2\lambda\rho}(y)f(x) = f(xy)$.

The set $\widehat{G}_{\mathrm{adm}}$ consists of

(a) the finite-dimensional representations δ_{2k}, $\det \otimes \delta_{2k}$ with $k \geq 0$ an integer, the dimension of δ_{2k} or $\det \otimes \delta_{2k}$ is $2k+1$,
(b) the induced representations $\pi_{\sigma,2\lambda\rho}$ with $\sigma \in \widehat{M}$, $\lambda \in \mathbb{C}$, $\lambda \notin \frac{1}{2} + \mathbb{Z}$,
(c) the discrete series representations $\mathcal{D}_{n+1}$ for $n \in \mathbb{N}$ odd.

The only isomorphisms between these are

$$\pi_{\sigma,2\lambda\rho} \cong \pi_{\sigma,-2\lambda\rho}, \quad \lambda \notin \frac{1}{2} + \mathbb{Z}.$$

For details, see [23]. For a given odd number $n \in \mathbb{N}$, there are exact sequences

$$0 \to \mathcal{D}_{n+1} \to \pi_{\sigma,n\rho} \to \sigma \otimes \delta_{n-1} \to 0,$$

where, by abuse of notation, we have written

$$\sigma \otimes \delta_{2k} = \begin{cases} \delta_{2k}, & \sigma = 1, \\ \det \otimes \delta_{2k}, & \sigma \neq 1, \end{cases}$$

and

$$0 \to \sigma \otimes \delta_{n-1} \to \pi_{\sigma,-n\rho} \to \mathcal{D}_{n+1} \to 0.$$

These $(\mathfrak{g}, K)$-morphisms are unique up to multiplication by scalars.

Definition 6.1. For $p \geq 0$, let $H^p(\mathfrak{n}, \pi)$ denote the Lie algebra cohomology of $\mathfrak{n} = \mathrm{Lie}_{\mathbb{C}}(N)$ with coefficients in $\pi \in \widehat{G}_{\mathrm{adm}}$ (see [3]). Note that, as $\dim \mathfrak{n} = 1$, the space $H^p(\mathfrak{n}, \pi)$ can be non-zero only if $p = 0$ or $p = 1$.

Definition 6.2. We write $\sigma \otimes \mathbb{C}_{2\lambda\rho}$ for the one-dimensional $\mathbb{C}[AM]$-module on which A acts via the morphism $2\lambda\rho$ and M acts via σ. As an abbreviation, we will sometimes write $\mathbb{C}_{2\lambda\rho}$ for $1 \otimes \mathbb{C}_{2\lambda\rho}$.

Proposition 6.3. *As AM-module, the Lie algebra cohomology* $H^p(\mathfrak{n}, \pi)$ *for* $\pi \in \widehat{G}_{\mathrm{adm}}$ *is isomorphic to*

- $H^0(\mathfrak{n}, \pi_{\sigma,2\lambda\rho}) = 0$ *for* $\lambda \in \mathbb{C}$ *except for* $\lambda = -\frac{n}{2}$ *for odd* $n \in \mathbb{N}$,
  $H^1(\mathfrak{n}, \pi_{\sigma,2\lambda\rho}) \cong (\sigma \det) \otimes \mathbb{C}_{2\lambda\rho-\rho} \oplus (\sigma \det) \otimes \mathbb{C}_{-2\lambda\rho-\rho}$ *if* $\lambda \notin \frac{1}{2} + \mathbb{Z}$,
- $H^0(\mathfrak{n}, \sigma \otimes \delta_{2k}) \cong (\sigma \det) \otimes \mathbb{C}_{2k\rho}$,
  $H^1(\mathfrak{n}, \sigma \otimes \delta_{2k}) \cong (\sigma \det) \otimes \mathbb{C}_{-2k\rho-2\rho}$,
- $H^0(\mathfrak{n}, \mathcal{D}_{n+1}) = 0$,
  $H^1(\mathfrak{n}, \mathcal{D}_{n+1}) \cong (\det \otimes \mathbb{C}_{n\rho-\rho}) \oplus \mathbb{C}_{n\rho-\rho}$.

Proof. Let $f \in H^0(\mathfrak{n}, \pi_{\sigma,2\lambda\rho})$. Then f is a smooth function $f\colon G \to \mathbb{C}$ satisfying $f(manx) = \sigma(m)a^{2\lambda\rho+\rho}f(x)$, $x \in G$, as well as $f(x\left[\begin{smallmatrix}1 & u \\ & 1\end{smallmatrix}\right]) = f(x)$ for every $u \in \mathbb{R}$. By means of the Iwasawa decomposition, we compute

$$f\left(\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}\right) = f\left(\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & u \\ & 1 \end{bmatrix}\right) = f\left(\frac{1}{\sqrt{1+u^2}}\begin{bmatrix} u & -1 \\ 1 & u \end{bmatrix}\right)(1+u^2)^{-\lambda-\frac{1}{2}}.$$

If $\mathrm{Re}(\lambda) > -\frac{1}{2}$, then the right-hand side tends to zero as $u \to \infty$. This gives $f(\left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right]) = 0$, which, again by the same equation, implies $f = 0$. The first claim therefore is proven for $\mathrm{Re}(\lambda) > -\frac{1}{2}$. As $\pi_{\sigma,2\lambda\rho} \cong \pi_{\sigma,-2\lambda\rho}$ if $\lambda \notin \frac{1}{2} + \mathbb{Z}$, the claim follows in general.

For $H^1(\mathfrak{n}, \pi_{\lambda\rho})$, we firstly use the Poincaré duality

$$H_p(\mathfrak{n}, \pi) \cong H^{1-p}(\mathfrak{n}, \pi) \otimes \det(\mathfrak{n}) \cong H^{1-p}(\mathfrak{n}, \pi) \otimes (\det \otimes \mathbb{C}_{2\rho}),$$

and secondly the Frobenius reciprocity

$$\mathrm{Hom}_{AM}(H_0(\mathfrak{n}, \pi), \sigma \otimes \mathbb{C}_{2\lambda\rho+\rho}) \cong \mathrm{Hom}_G(\pi, \pi_{\sigma,2\lambda\rho}).$$

These two are standard and can be found in [3] or in [20], for instance. By the classification of $\widehat{G}_{\mathrm{adm}}$, we have $\mathrm{Hom}_G(\pi_{\sigma,2\lambda\rho}, \pi) \neq 0$ if and only if $\pi = \pi_{\sigma,\pm 2\lambda\rho}$ and, in this case, we derive from the Frobenius reciprocity that

$$\mathrm{H}_0(\mathfrak{n}, \pi_{\sigma,2\lambda\rho}) \cong (\sigma \otimes \mathbb{C}_{2\lambda\rho+\rho}) \oplus (\sigma \otimes \mathbb{C}_{-2\lambda\rho+\rho}).$$

By the Poincaré duality, we get the claim.

The representation δ_{2k} can be realized as representation on the space of all polynomials $p(X, Y)$ in two variables which are homogeneous of degree $2k$. A basis of this space is $X^{2k}, X^{2k-1}Y, \dots, Y^{2k}$. The representation is $\delta_{2k}(g)p(X, Y) = p((X, Y)g)$. It emerges that $\mathrm{H}^0(\mathfrak{n}, \sigma \otimes \delta_{2k})$ is spanned by $1 \otimes X^{2k}$ and, since AM acts on this by $\sigma \otimes 2k\rho$, we get $\mathrm{H}^0(\mathfrak{n}, \sigma \otimes \delta_{2k}) \cong \sigma \otimes \mathbb{C}_{2k\rho}$. Further, as $\sigma \otimes \delta_{2k}$ is a sub-representation of $\pi_{\sigma,-2k\rho-\rho}$, the Frobenius reciprocity yields $\mathrm{H}_0(\mathfrak{n}, \sigma \otimes \delta_{2k}) \cong \sigma \otimes \mathbb{C}_{-2k\rho}$ so that again the Poincaré duality gives the claim.

As $\mathcal{D}_{n+1}$ is a sub-representation of $\pi_{\sigma,n\rho}$, the fact $\mathrm{H}^0(\mathfrak{n}, \mathcal{D}_{n+1}) = 0$ follows from $\mathrm{H}^0(\mathfrak{n}, \pi_{\sigma,n\rho}) = 0$. Finally, $\mathcal{D}_{n+1}$ is a sub-representation of $\pi_{\sigma,n\rho}$ in essentially one way, so the Frobenius reciprocity yields

$$\mathrm{H}_0(\mathfrak{n}, \mathcal{D}_{n+1}) \cong \mathbb{C}_{n\rho+\rho} \oplus (\det \otimes \mathbb{C}_{n\rho+\rho}),$$

and the claim follows by duality. □

Definition 6.4. For a complex vector space V on which AM acts linearly, let $V^{\sigma}_{2\lambda\rho}$ be the space of all $v \in V$ with $(am - \sigma(m)a^{2\lambda\rho})^n v = 0$ for some $n \in \mathbb{N}$ and all $am \in AM$.

For $\sigma \in \widehat{M}$, $\lambda \in \mathfrak{a}^*$ and $\pi \in \widehat{G}_{\mathrm{adm}}$, define the *spectral Lefschetz numbers* by

$$L^{\sigma}_{2\lambda\rho}(\pi) = \sum_{q=0}^{\dim \mathfrak{n}} (-1)^{q+\dim \mathfrak{n}} \dim \mathrm{H}^q(\mathfrak{n}, \pi)^{\sigma}_{2\lambda\rho}$$

Corollary 6.5. *The computations in Proposition 6.3 yield*

$$L^{\sigma}_{2\lambda\rho}(1) = \begin{cases} 1, & \lambda = -1, \\ -1, & \sigma = \det, \lambda = 0, \\ 0 & \textit{otherwise}. \end{cases}$$

$$L^{\sigma}_{2\lambda\rho}(\det) = \begin{cases} 1, & \sigma = 1, \lambda = -1, \\ -1, & \sigma = 1, \lambda = 0, \\ 0 & \textit{otherwise}. \end{cases}$$

$$L^{\sigma}_{2\lambda\rho}(\pi_{\sigma_1,2\mu\rho}) = \begin{cases} 1, & \sigma = \sigma_1 \det, \lambda = \pm\mu - \frac{1}{2}, \\ 0 & \textit{otherwise}. \end{cases}$$

$$L^{\sigma}_{2\lambda\rho}(\mathcal{D}_{n+1}) = \begin{cases} 1, & \lambda = \frac{n-1}{2}, \\ 0 & \textit{otherwise}. \end{cases}$$

Next, for the geometric Lefschetz numbers, if $[\gamma] \in \mathcal{E}_P(\Gamma)$, then γ is G-conjugate to $\begin{bmatrix} N(\gamma)^{1/2} & \\ & N(\gamma)^{-1/2} \end{bmatrix}$ for some $N(\gamma) > 1$. An element γ of Γ is called *primitive* if $\gamma = \sigma^n$ for $\sigma \in \Gamma$ and $n \in \mathbb{N}$ implies $n = 1$. Each $\gamma \in \mathcal{E}_P(\Gamma)$ is a power of a unique primitive γ_0 which will be called the primitive *underlying* γ. Then

$$L(\gamma) = \frac{\log N(\gamma_0)}{1 - N(\gamma)^{-1}}.$$

We will now recall some facts about the *Selberg zeta function* [14, 21, 22]. Let $\mathcal{E}^p_P(\Gamma)$ denote the set of all primitive classes in $\mathcal{E}_P(\Gamma)$. The Selberg zeta function is given by the product

$$Z(s) = \prod_{\gamma \in \mathcal{E}^p_P(\Gamma)} \prod_{k=0}^{\infty} (1 - N(\gamma)^{-s-k}).$$

The product converges locally uniformly for $\mathrm{Re}(s) > 1$. The zeta function extends to a meromorphic function on the plane of finite order. It has a simple zero at $s = 1$ and zeros at $s = \frac{1}{2} \pm u$ of multiplicity $N_\Gamma(\pi_{u\frac{\rho}{2}})$. These are all zeros or poles in $\mathrm{Re}(s) \geq \frac{1}{2}$ except for $s = \frac{1}{2}$ where $Z(s)$ has a zero or pole of order $N_\Gamma(\pi_0)$ minus the number of cusps. The poles and zeros in $\mathrm{Re}(s) < \frac{1}{2}$ can be described through the scattering matrix or intertwining operators [14, 21, 22].

Recall the inversion formula for the Mellin transform. Let the function ψ be integrable on $(0, \infty)$ with respect to the measure $\frac{dt}{t}$, in other words, $\psi \in L^1((0,\infty), \frac{dt}{t})$. Then the *Mellin transform* of ψ is given by

$$M\psi(s) = \int_0^\infty t^s \psi(t) \frac{dt}{t}, \quad s \in i\mathbb{R}.$$

If the function ψ is continuously differentiable and $\psi'(t)t$, $\psi''(t)t^2$ are also in $L^1((0,\infty), \frac{dt}{t})$, then the following inversion formula holds:

$$\psi(t) = \frac{1}{2\pi i} \int_{i\mathbb{R}} M\psi(s) t^{-s}\, ds.$$

Now assume that ψ is supported in the interval $[1, \infty)$ and, for some $\mu > 0$, the functions $\psi(t)$, $\psi'(t)t$, $\psi''(t)t^2$ all are $O(t^{-\mu})$. Then it follows that the integral $M\psi(s)$ defines a function holomorphic in $\mathrm{Re}(s) < \mu$, and the integral in the inversion formula can be shifted,

$$\psi(t) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} M\psi(s) t^{-s}\, ds \quad \text{for every } C < \mu.$$

Every $\gamma \in \mathcal{E}_P(\Gamma)$ can be written as $\gamma = \gamma_0^n$ for some uniquely determined $\gamma_0 \in \mathcal{E}_P^p(\Gamma)$ and a unique $n \in \mathbb{N}$. A computation yields for $\mathrm{Re}(s) > 1$,

$$\frac{Z'}{Z}(s) = \sum_{\gamma \in \mathcal{E}_P^p(\Gamma)} \sum_{n=1}^\infty \frac{\log N(\gamma)}{1 - N(\gamma)^{-n}} N(\gamma)^{-ns} = \sum_{\gamma \in \mathcal{E}_P(\Gamma)} \frac{\log N(\gamma_0)}{1 - N(\gamma)^{-1}} N(\gamma)^{-s}.$$

Let ψ be as above with $\mu > 1$, and let $1 < C < \mu$. Then, since $\frac{Z'}{Z}(s)$ is bounded in $\mathrm{Re}(s) = C$, we can interchange integration and summation to get

$$\frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \frac{Z'}{Z}(s) M\psi(s)\, ds = \sum_{\gamma \in \mathcal{E}_P(\Gamma)} \frac{\log N(\gamma_0)}{1 - N(\gamma)^{-1}} \psi(N(\gamma)).$$

For $a \in A^- = \{\left[\begin{smallmatrix} t & \\ & t^{-1}\end{smallmatrix}\right] : 0 < t < 1\}$, set

$$\phi(a) = \phi\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix} = \psi\left(\frac{1}{t}\right).$$

Then $\phi \in C^{2,2\mu\rho}(A^-)$ and

$$\frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \frac{Z'}{Z}(s) M\psi(s)\, ds = \sum_{\gamma \in \mathcal{E}_P(\Gamma)} \frac{\log N(\gamma_0)}{1 - N(\gamma)^{-1}} \phi(a_\gamma) = \sum_{\gamma \in \mathcal{E}_P(\Gamma)} L(\gamma)\phi(a_\gamma),$$

which is the right-hand side of the Lefschetz formula.

Now suppose that $\phi \in C^{j,2\mu\rho}(A^-)$ for some $j \in \mathbb{N}$ and $\mu > 1$. Then the functions $\psi(t)$, $\psi'(t)t, \dots, \psi^{(j)}(t)t^j$ are all $O(t^{-\mu})$. Integration by parts shows that

$$M\psi(t) \frac{(-1)^j}{s(s+1)\cdots(s+j-1)} \int_0^\infty t^s \psi^{(j)}(t) t^j \frac{dt}{t}.$$

This implies $M\psi(s) = O((1+|s|)^{-j})$ uniformly in $\{\mathrm{Re}(s) \leq \alpha\}$ for every $\alpha < \mu$.

For $R > 0$ and $a \in \mathbb{C}$, let $B_r(a)$ be the closed disk around a of radius r. Let g be a meromorphic function on $\mathbb{C}$ with poles $a_1, a_2, \dots$. We say that g is *essentially of moderate growth* if there is a natural number N, a constant $C > 0$ and a sequence of real numbers $r_n > 0$ tending to zero such that the disks $B_{r_n}(A_n)$ are pairwise disjoint and that on the domain $D = \mathbb{C} \smallsetminus \bigcup_n B_{r_n}(a_n)$ it holds $|g(z)| \leq C|z|^N$. Every such N is called a *growth exponent* of g.

Lemma 6.6. *Let h be a meromorphic function on* $\mathbb{C}$ *of finite order, and let* $g = h'/h$ *be its logarithmic derivative. Then g is essentially of moderate growth with growth exponent equal to the order of h plus two.*

Proof. This is a direct consequence of Hadamard's factorization theorem applied to h. □

This lemma together with the growth estimate for $M\psi$ implies that, for j large enough, the contour integral over $C + i\mathbb{R}$ can be moved to the left, deforming it slightly, so that one stays in the domain D and gathering residues. Ultimately, the contour integral will tend to zero, leaving only the residues. One gets

$$\begin{aligned}\sum_{\gamma\in\mathcal{E}_P(\Gamma)} L(\gamma)\phi(a_\gamma) &= \sum_{s_0\in\mathbb{C}}\Big(\operatorname{res}_{s=s_0}\frac{Z'}{Z}(s)\Big)M\psi(s_0)\\ &= \sum_{s_0\in\mathbb{C}}\Big(\operatorname{res}_{s=s_0}\frac{Z'}{Z}(s)\Big)\int_0^\infty \psi(t)t^{s_0}\frac{dt}{t}\\ &= \sum_{s_0\in\mathbb{C}}\Big(\operatorname{res}_{s=s_0}\frac{Z'}{Z}(s)\Big)\int_{A^-}\phi(a)a^{-s_0\rho}\,da.\end{aligned}$$

This implies the conjecture in the case $G = \mathrm{PGL}_2(\mathbb{R})$.

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