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# Selberg and Ruelle zeta functions for non-unitary twists

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**Abstract** In this paper we study the dynamical zeta functions of Ruelle and Selberg associated with the geodesic flow of a compact hyperbolic odd-dimensional manifold. These are functions of a complex variable $s$ in some right half-plane of $\mathbb{C}$. Using the Selberg trace formula for arbitrary finite dimensional representations of the fundamental group of the manifold, we establish the meromorphic continuation of the dynamical zeta functions to the whole complex plane. We explicitly describe the singularities of the Selberg zeta function in terms of the spectrum of certain twisted Laplace and Dirac operators.

**Keywords** Selberg zeta function · Ruelle zeta function · Selberg trace formula · Bochner–Laplace operator · Dirac operator

## 1 Introduction

The Selberg and Ruelle zeta functions, studied in this paper, are dynamical zeta functions, which are associated with the geodesic flow on the unit sphere bundle $S(X)$ of a compact hyperbolic manifold $X$. Namely, they are defined in terms of the lengths of the closed geodesics, also called length spectrum. The dynamical zeta functions are defined by Euler-type products, which converge in some right half-plane of $\mathbb{C}$. The main goal of this paper is to prove the meromorphic continuation of these functions to the whole complex plane.

The Ruelle zeta function associated with the geodesic flow on the unit sphere bundle of a closed manifold with $C^{\omega}$-Riemannian metric of negative curvature has been studied by Fried in [7]. It is defined by

$$R(s) = \prod_{\gamma}(1 - e^{-sl(\gamma)}),$$

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where $\gamma$ runs over all the prime closed geodesics and $l(\gamma)$ denotes the length of $\gamma$. In [7, Corollary, p. 180], it is proved that it admits a meromorphic continuation to the whole complex plane.

We consider an oriented compact hyperbolic manifold $X$ of odd dimension $d$, obtained as follows. Let either $G = \mathrm{SO}^0(d,1)$, $K = \mathrm{SO}(d)$ or $G = \mathrm{Spin}(d,1)$, $K = \mathrm{Spin}(d)$. Then, $K$ is a maximal compact subgroup of $G$. Let $\widetilde{X} := G/K$. $\widetilde{X}$ can be equipped with a $G$-invariant metric, which is unique up to scaling and is of constant negative curvature. If we normalize this metric such that it has constant negative curvature $-1$, then $\widetilde{X}$, equipped with this metric, is isometric to the $d$-dimensional hyperbolic space $\mathbb{H}^d$. Let $\Gamma$ be a discrete torsion-free subgroup of $G$ such that $\Gamma\backslash G$ is compact. Then, $\Gamma$ acts by isometries on $\widetilde{X}$ and $X = \Gamma\backslash\widetilde{X}$ is a compact oriented hyperbolic manifold of dimension $d$. This is a case of a locally symmetric space of non-compact type of real rank 1. This means that in the Iwasawa decomposition $G = KAN$, $A$ is a multiplicative torus of dimension 1, i.e., $A \cong \mathbb{R}^+$.

For a given $\gamma \in \Gamma$ we denote by $[\gamma]$ the $\Gamma$-conjugacy class of $\gamma$. If $\gamma \neq e$, then there is a unique closed geodesic $c_\gamma$ associated with $[\gamma]$. Let $l(\gamma)$ denote the length of $c_\gamma$. The conjugacy class $[\gamma]$ is called primitive if there exist no $k \in \mathbb{N}$ with $k>1$ and $\gamma_0 \in \Gamma$ such that $\gamma = \gamma_0^k$. The prime closed geodesics correspond to the primitive conjugacy classes and are those geodesics that trace out their image exactly once.

Let $M$ be the centralizer of $A$ in $K$. Since $\Gamma$ is a cocompact subgroup of $G$, every element $\gamma \in \Gamma - \{e\}$ is hyperbolic. Then, by [31, Lemma 6.5], there exist a $g \in G$, a $m_\gamma \in M$, and an $a_\gamma \in A$, such that $g^{-1}\gamma g = m_\gamma a_\gamma$. The element $m_\gamma$ is determined up to conjugacy in $M$, and the element $a_\gamma$ depends only on $\gamma$. Let $\mathfrak{g}$, $\mathfrak{a}$ be the Lie algebras of $G$ and $A$, respectively. We define the zeta functions associated with unitary irreducible representations $\sigma$ of $M$ and finite dimensional representations $\chi$ of $\Gamma$. The twisted Selberg zeta function $Z(s;\sigma,\chi)$ is defined for $s \in \mathbb{C}$ by the infinite product

$$Z(s;\sigma,\chi) := \prod_{\substack{[\gamma]\neq e,\\ [\gamma]\,\text{prime}}} \prod_{k=0}^{\infty} \det\Big(\mathrm{Id} - \big(\chi(\gamma)\otimes\sigma(m_\gamma)\otimes S^k(\mathrm{Ad}(m_\gamma a_\gamma)|_{\overline{\mathfrak{n}}})\big)e^{-(s+|\rho|)l(\gamma)}\Big),$$

where $\overline{\mathfrak{n}}$ is the sum of the negative root spaces of $(\mathfrak{g},\mathfrak{a})$, $S^k(\mathrm{Ad}(m_\gamma a_\gamma)|_{\overline{\mathfrak{n}}})$ denotes the $k$th symmetric power of the adjoint map $\mathrm{Ad}(m_\gamma a_\gamma)$ restricted to $\overline{\mathfrak{n}}$ and $\rho := \frac{1}{2}\dim(\mathfrak{g}_\alpha)\alpha$ (see Sect. 2).

The twisted Ruelle zeta function $R(s;\sigma,\chi)$ is defined for $s \in \mathbb{C}$ by the infinite product

$$R(s;\sigma,\chi) := \prod_{\substack{[\gamma]\neq e\\ [\gamma]\,\text{prime}}} \det\Big(\mathrm{Id} - \chi(\gamma)\otimes\sigma(m_\gamma)e^{-sl(\gamma)}\Big)^{(-1)^{d-1}}.$$

Both $Z(s;\sigma,\chi)$ and $R(s;\sigma,\chi)$ converge absolutely and uniformly on compact subsets of some half-plane of $\mathbb{C}$.

In our case the dynamical zeta functions are twisted by a representation $\chi$ of $\Gamma$. For unitary representations of $\Gamma$, these zeta functions have been studied by Fried [6] and Bunke and Olbrich [2]. In [6, Theorem 1], Fried proved that the Ruelle zeta function on a closed oriented hyperbolic manifold associated with an acyclic orthogonal representation of the fundamental group admits a meromorphic continuation to the whole complex plane, and furthermore, the absolute value of the Ruelle zeta function evaluated at zero equals the Ray–Singer analytic torsion as it is introduced in [25]. The theorem of Fried gave rise to other important applications in the field of spectral geometry, such as the asymptotic behavior of the analytic torsion [21] and the study of the growth of the torsion in the cohomology of a closed arithmetic hyperbolic 3-manifold [17].

Müller [21] considered the asymptotic behavior of the analytic torsion of a closed hyperbolic 3-manifold $X$, associated with the $m$th symmetric power $\tau_m$ of the standard representation of $\mathrm{SL}(2,\mathbb{C})$. In particular, he proved that for a closed hyperbolic oriented 3-manifold,

$$-\log T_X(\tau_m) = \frac{\mathrm{Vol}(X)}{4\pi} m^2 + O(m),$$

as $m \to \infty$ [21, Theorem 1.1]. In order to prove this theorem, he used the expression of the analytic torsion in terms of the Ruelle zeta function attached to $\tau_m$ and based on the results of Wotzke [33]. In his thesis, Wotzke generalized the theorem of Fried to representations, which are obtained as restrictions to $\Gamma$ of a finite dimensional complex representation $\tau\colon G \to \mathrm{GL}(V)$ of $G$. By [18, Proposition 3.1], there exists an isomorphism between the locally homogeneous vector bundle $E_\tau$ over $X$ associated with $\tau|_K$ and the flat vector bundle $E_{\mathrm{flat}}$ over $X$ associated with $\tau|_\Gamma$, i.e.,

$$(\Gamma\backslash G \times V)/K \cong \Gamma\backslash(G/K \times V).$$

Once this isomorphism is obtained, a Hermitian fiber metric in $E_\tau$ [18, Lemma 3.1] descends to a fiber metric in $E_{\mathrm{flat}}$. Therefore, the Laplace operator associated with $\tau|_\Gamma$ is a formally self-adjoint operator and it can be studied by methods of harmonic analysis.

On the other hand, Bunke and Olbrich [2] consider a unitary representation of the fundamental group for all cases of compact locally symmetric spaces of real rank 1 of non-compact type, i.e., the compact manifolds whose universal coverings are the real, complex, or quaternionic hyperbolic space, or the Cayley plane. Using the Selberg trace formula as main tool for wave operators induced by certain Laplace-type and Dirac operators, they proved that the Selberg and Ruelle zeta functions admit a meromorphic continuation to the whole complex plane and furthermore satisfy functional equations.

Our main aim is to generalize the results of Bunke and Olbrich to the case of non-unitary representations of the fundamental group. Contrary to the setting of Wotzke, we can no longer use the results in [18], since we treat the case of an arbitrary finite dimensional representation $\chi\colon \Gamma \to \mathrm{GL}(V_\chi)$. Let $E_\chi$ be the flat vector bundle over $X$, associated with $\chi$, equipped with a flat connection $\nabla^\chi$. In general, there is no Hermitian metric $h^\chi$, which is compatible with the flat connection. To overcome this problem we use the twisted Bochner–Laplace operator, first introduced by Müller [23]. This operator is defined as follows.

Let $\tau\colon K \to \mathrm{GL}(V_\tau)$ be a finite dimensional unitary representation of $K$. Let $\widetilde{E}_\tau := G \times_\tau V_\tau \to \widetilde{X}$ be the associated homogeneous vector bundle over $\widetilde{X}$. Let $E_\tau := \Gamma\backslash(G \times_\tau V_\tau) \to X$ be the locally homogeneous vector bundle over $X$. Let $\Delta_\tau$ be the Bochner–Laplace operator associated with $\tau$. We define the twisted Bochner–Laplace operator $\Delta^\sharp_{\tau,\chi}$ acting on $C^\infty(X, E_\tau \otimes E_\chi)$. If we consider the lifts $\widetilde{\Delta}^\sharp_{\tau,\chi}$ and $\widetilde{\Delta}_\tau$ to $\widetilde{X}$ of $\Delta^\sharp_{\tau,\chi}$ and $\Delta_\tau$, respectively, then,

$$\widetilde{\Delta}^\sharp_{\tau,\chi} = \widetilde{\Delta}_\tau \otimes \mathrm{Id}_{V_\chi}.$$

Our operator is not self-adjoint anymore. However, it has the same principal symbol as $\Delta_\tau \otimes \mathrm{Id}_{V_\chi}$ and hence has nice spectral properties, i.e., its spectrum is discrete and contained in a translate of a positive cone in $\mathbb{C}$. We consider the corresponding heat semigroup $e^{-t\Delta^\sharp_{\tau,\chi}}$ acting on the space of smooth sections of the vector bundle $E_\tau \otimes E_\chi$. By [23, Lemma 2.4], it is an integral operator with smooth kernel. Hence, we can consider the trace of the operator $e^{-t\Delta^{\sharp}_{\tau,\chi}}$ and derive a corresponding trace formula.

We already associated the Selberg and Ruelle zeta functions with irreducible representations $\sigma$ of $M$. These representations are chosen precisely to be the representations arising from restrictions of representations of $K$. Let $M'$ be the normalizer of $A$ in $K$. Let $W_A$ be the restricted Weyl group defined by $W_A := M'/M$. For $m \in M$ and $\sigma \in \widehat{M}$, the action of $W_A$ on $\widehat{M}$ is defined by $(w\sigma)(m) := \sigma(m_w^{-1} m m_w)$, where $w \in W_A$ is the non-trivial element of $W_A$, and $m_w$ a representative of $w$ in $M'$. Then, $\sigma$ is Weyl invariant if $w\sigma = \sigma$. In general, $\sigma$ can be non-Weyl invariant, i.e., $w\sigma \neq \sigma$. In such a case we have to introduce three more zeta functions: the symmetrized $S(s, \sigma, \chi)$, super $Z^s(s, \sigma, \chi)$, and super Ruelle $R^s(s\, \sigma, \chi)$ zeta function (Definitions 3.6, 3.7 and 3.8, respectively). To be able to prove the meromorphic continuation of the above zeta functions, we introduce the twisted Dirac operator, defined in a similar way as the twisted Bochner–Laplace operator in [23].

We give here a short description of the twisted Dirac operator. Let $\sigma \in \widehat{M}$. Let $\widehat{K}$ be the set of the equivalent classes of unitary irreducible representations of $K$. Let $\tau(\sigma)$ be the representation in $\widehat{K}$, associated with $\sigma$ (for further details we refer the reader to Sect. 5, Proposition 5.4). Let $s$ be the spin representation of $K$. We define the representation $\tau_s(\sigma)$ of $K$ by $\tau_s(\sigma) := s \otimes \tau(\sigma)$. We consider the locally homogeneous vector bundle $E_{\tau_s(\sigma)}$ over $X$. We let $D(\sigma)$ be the Dirac operator associated with $\tau_s(\sigma)$. Let $D^{\sharp}_{\chi}(\sigma)$ be the twisted Dirac operator acting on $C^{\infty}(X, E_{\tau_s(\sigma)} \otimes E_{\chi})$. The lift $\widetilde{D}^{\sharp}_{\chi}(\sigma)$ of $D^{\sharp}_{\chi}(\sigma)$ to the universal covering $\widetilde{X}$ is given by the following formula

$$\widetilde{D}^{\sharp}_{\chi}(\sigma) = \widetilde{D}(\sigma) \otimes \mathrm{Id}_{V_{\chi}},$$

where $\widetilde{D}(\sigma)$ is the lift of $D(\sigma)$ to $\widetilde{X}$. The twisted Dirac operator $D^{\sharp}_{\chi}(\sigma)$ has the same principal symbol as $D(\sigma) \otimes \mathrm{Id}_{V_{\chi}}$.

In Sect. 6.2, we extend the Parthasarathy formula [2, Eq. (1.11)] to the case of the non-unitary twist, i.e., $D^{\sharp}_{\chi}(\sigma)^2 = A^{\sharp}_{\chi}(\sigma)$. Here $A^{\sharp}_{\chi}(\sigma)$ is the auxiliary operator defined as in [2, p. 28]. In our case, this operator is twisted by a non-unitary representation $\chi$. For more details we refer the reader to Sect. 5. Let $D^{\sharp}_{\chi}(\sigma)e^{-tD^{\sharp}_{\chi}(\sigma)^2}$ be the associated operator acting on the space of smooth sections of the vector bundle $E_{\tau_s(\sigma)} \otimes E_{\chi}$. By [23, Lemma 2.4], this is an integral operator with smooth kernel and hence of trace class. We derive a corresponding trace formula for $D^{\sharp}_{\chi}(\sigma)e^{-tD^{\sharp}_{\chi}(\sigma)^2}$. Our main results are stated in the following theorems.

**Theorem 1.1** *Let $\sigma \in \widehat{M}$ and $\chi : \Gamma \to \mathrm{GL}(V_{\chi})$ be a finite dimensional representation of $\Gamma$. The twisted Selberg zeta function $Z(s; \sigma, \chi)$ associated with $\sigma$ and $\chi$ admits a meromorphic continuation to the whole complex plane $\mathbb{C}$. Its singularities are described in terms of the discrete eigenvalues of the twisted operators $A^{\sharp}_{\chi}(\sigma)$ and $D^{\sharp}_{\chi}(\sigma)$, and their orders are described by the corresponding algebraic multiplicities.*

**Theorem 1.2** *For every $\sigma \in \widehat{M}$ and for every finite dimensional representation $\chi$ of $\Gamma$, the twisted Ruelle zeta function $R(s; \sigma, \chi)$ admits a meromorphic continuation to the whole complex plane $\mathbb{C}$.*

The keypoint for the proof of the meromorphic continuation of the Selberg zeta function is the trace formulas for the operators

- $e^{-tA^{\sharp}_{\chi}(\sigma)}$, if $\sigma$ is Weyl invariant;
- $D^{\sharp}_{\chi}(\sigma)e^{-tD^{\sharp}_{\chi}(\sigma)^2}$, if $\sigma$ is non-Weyl invariant.

The resolvent operator $(A_\chi(\sigma)+s^2)^{-1}$ is related to the corresponding heat operator by

$$\left(A^\sharp_\chi(\sigma)+s^2\right)^{-1}=\int_0^\infty e^{-ts^2}e^{-tA^\sharp_\chi(\sigma)}\mathrm{d}t, \tag{1.1}$$

for $\mathrm{Re}(s^2)>-c$, where $c$ is a real number such that $\operatorname{spec}\left(A^\sharp_\chi(\sigma)\right)\subset\{z\in\mathbb{C}\colon \mathrm{Re}(z)>c\}$.

A similar identity holds for the resolvent operator $D^\sharp_\chi(\sigma)(D^\sharp_\chi(\sigma)^2+s^2)^{-1}$. We use the generalized resolvent identity, as it is described in [2, Lemma 3.5], to consider a finite product $\prod_{i=1}^N(A^\sharp_\chi(\sigma)+s_i^2)^{-1}$ and apply the trace formula on the right-hand side of (1.1). The *spectral* singularities of the Selberg zeta function are arising from the poles of the associated resolvent operator. The trace formulas we employ are stated in Sects. 5 and 6 (Sect. 6.2) for the operators $e^{-tA^\sharp_\chi(\sigma)}$ and $D^\sharp_\chi(\sigma)e^{-tD^\sharp_\chi(\sigma)^2}$, respectively. If we insert the right-hand side of the trace formulas for the operators $P_t=e^{-tA^\sharp_\chi(\sigma)}$ or $D^\sharp_\chi(\sigma)e^{-tD^\sharp_\chi(\sigma)^2}$ in the integral

$$\int_0^\infty\sum_{i=1}^N\left(\prod_{\substack{j=1\\ j\neq i}}^N\frac{1}{s_j^2-{s_i}^2}\right)e^{-ts_i^2}\,\mathrm{Tr}\;P_t\mathrm{d}t,$$

then the integral that includes the hyperbolic contribution to the trace formula for $P_t$ is related to

- $\frac{\mathrm{d}}{\mathrm{d}s}\log Z(s;\sigma,\chi)$, if $P_t=e^{-tA^\sharp_\chi(\sigma)}$ and $\sigma$ is Weyl invariant;
- $\frac{\mathrm{d}}{\mathrm{d}s}\log S(s;\sigma,\chi)$, if $P_t=e^{-tA^\sharp_\chi(\sigma)}$ and $\frac{\mathrm{d}}{\mathrm{d}s}\log Z^s(s;\sigma,\chi)$, if $P_t=D^\sharp_\chi(\sigma)e^{-tD^\sharp_\chi(\sigma)^2}$, and $\sigma$ is non-Weyl invariant.

Further, the proof for the meromorphic continuation of the Ruelle zeta function is based on the expression of the Ruelle zeta function as a product of Selberg zeta functions with shifted origins, as it is stated in Theorem 7.12 in Sect. 7.5. The trace formula unifies the treatment of different cases and gives rise to functional equations for the dynamical zeta functions [28].

This paper is organized as follows. Basic notions and facts are summarized in Sect. 2. In Sect. 3, we introduce the twisted Selberg and Ruelle zeta functions associated with the geodesic flow of a compact hyperbolic manifold and prove that they converge in some right half-plane of $\mathbb{C}$ for an arbitrary finite dimensional representation of $\Gamma$. Section 4 is devoted to the study of the twisted Bochner–Laplace operator on a compact hyperbolic odd-dimensional manifold. Next, in Sect. 5, we define the auxiliary operator $A^\sharp_\chi(\sigma)$ and the corresponding heat operator $e^{-tA^\sharp_\chi(\sigma)}$. The definitions of the homogeneous vector bundles associated with the representations $\sigma$ and $\chi$ of the groups $M$ and $\Gamma$, respectively, are provided. At the end of the section, we prove the trace formula for the operator $e^{-tA^\sharp_\chi(\sigma)}$. In Sect. 6, we introduce the twisted Dirac operator on $X$ and explain its relation to the twisted Bochner–Laplace operator. Moreover, we derive the trace formula for the integral operator $D^\sharp_\chi(\sigma)e^{-tD^\sharp_\chi(\sigma)^2}$. Finally, in Sect. 7, we consider resolvent identities and complete the proof for the meromorphic continuation of the dynamical zeta functions to the whole complex plane. We include also an "Appendix" (Sect. 8) to summarize the spectral properties of the twisted operators.

# 2 Preliminaries

## 2.1 Odd-dimensional hyperbolic manifolds

A special case of a locally symmetric space of real rank 1 is a compact hyperbolic manifold with universal covering the real hyperbolic space

$$\mathbb{H}^d = \left\{(x_1, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} : x_1^2 - x_2^2 \ldots - x_{d+1}^2 = 1, x_1 > 0\right\},$$

where $d = 2n + 1$, and $n \in \mathbb{N}_{>0}$. We consider the universal coverings $G = \mathrm{Spin}(d, 1)$ of $\mathrm{SO}^0(d, 1)$ and $K = \mathrm{Spin}(d)$ of $\mathrm{SO}(d)$, respectively. We set $\widetilde{X} := G/K$. Let $\mathfrak{g}, \mathfrak{k}$ be the Lie algebras of $G$ and $K$, respectively. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. We denote by $\Theta$ the Cartan involution of $G$ and by $\theta$ the differential of $\Theta$ at $e$, which is the identity element of $G$. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$. There exists a canonical isomorphism $T_{eK} \cong \mathfrak{p}$. We consider the subgroup $A$ of $G$ with Lie algebra $\mathfrak{a}$. Let $M := \mathrm{Centr}_K(A)$ be the centralizer of $A$ in $K$. Then, $M = \mathrm{Spin}(d - 1)$. Let $\mathfrak{m}$ be its Lie algebra and $\mathfrak{b}$ a Cartan subalgebra of $\mathfrak{m}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We consider the complexifications $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \oplus i\mathfrak{g}$, $\mathfrak{h}_{\mathbb{C}} := \mathfrak{h} \oplus i\mathfrak{h}$ and $\mathfrak{m}_{\mathbb{C}} := \mathfrak{m} \oplus i\mathfrak{m}$. Let $B(X, Y)$ be the Killing form on $\mathfrak{g} \times \mathfrak{g}$ defined by $B(X, Y) = \mathrm{Tr}\,(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$. It is a symmetric bilinear form. We consider the symmetric bilinear form

$$\langle Y_1, Y_2\rangle := \frac{1}{2(d-1)} B(Y_1, Y_2), \quad Y_1, Y_2 \in \mathfrak{g}. \tag{2.1}$$

The restriction of $\langle\cdot,\cdot\rangle$ to $\mathfrak{p}$ defines an inner product on $\mathfrak{p}$ and therefore induces a $G$-invariant Riemannian metric on $\widetilde{X}$, which has constant curvature $-1$. Then, $\widetilde{X}$ equipped with this metric is isometric to $\mathbb{H}^d$.

Let $\Gamma_1$ be a torsion-free cocompact discrete subgroup of $\mathrm{SO}^0(d, 1)$. We assume that $\Gamma_1$ can be lifted to a subgroup $\Gamma$ of $G$. Then, $X := \Gamma\backslash\widetilde{X}$ is a compact hyperbolic manifold of odd dimension $d$.

## 2.2 Representation theory of Lie groups

Let $G = KAN$ be the standard Iwasawa decomposition of $G$. Let $\Delta^+(\mathfrak{g}, \mathfrak{a})$ be the set of positive roots of $(\mathfrak{g}, \mathfrak{a})$ and $\mathfrak{g}_\alpha$ the corresponding root spaces. In the present case $\Delta^+(\mathfrak{g}, \mathfrak{a})$ consists of a single root $\alpha$. Let $M' = \mathrm{Norm}_K(A)$ be the normalizer of $A$ in $K$. We define the restricted Weyl group as the quotient $W_A := M'/M$. Then, $W_A$ has order 2. Let $w \in W_A$ be the non-trivial element of $W_A$, and $m_w$ a representative of $w$ in $M'$. The action of $W_A$ on $\widehat{M}$ is defined by

$$(w\sigma)(m) := \sigma\left(m_w^{-1} m m_w\right), \quad m \in M, \sigma \in \widehat{M}.$$

We will distinguish the following two cases:

- $\sigma$ is invariant under the action of the restricted Weyl group $W_A$, i.e., $w\sigma = \sigma$;
- $\sigma$ is not invariant under the action of the restricted Weyl group $W_A$, i.e., $w\sigma \neq \sigma$.

Let $H \in \mathfrak{a}$ such that $\alpha(H) = 1$. With respect to the inner product, induced by (2.1) on $\mathfrak{a}$, $H$ has norm 1. We define

$$A^+ := \left\{\exp(tH) : t \in \mathbb{R}^+\right\}. \tag{2.2}$$

We define also

$$\rho := \frac{1}{2}\dim(\mathfrak{g}_\alpha)\alpha, \tag{2.3}$$

$$\rho_{\mathfrak{m}} := \frac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{m}_{\mathbb{C}},\mathfrak{b})}\alpha. \tag{2.4}$$

The inclusion $i\colon M \hookrightarrow K$ induces the restriction map $i^*\colon R(K) \to R(M)$, where $R(K)$, $R(M)$ are the representation rings over $\mathbb{Z}$ of $K$ and $M$, respectively. Let $\widehat{K}$, $\widehat{M}$ be the sets of equivalent classes of irreducible unitary representations of $K$ and $M$, respectively. For the highest weight $\nu_\tau$ of $\tau \in \widehat{K}$ we have $\nu_\tau = (\nu_1, \ldots, \nu_n)$, where $\nu_1 \geq \cdots \geq \nu_n$ and $\nu_i$, $i = 1, \ldots, n$ are all half integers (that is $\nu_i = q_i + \frac{1}{2}$, $q_i \in \mathbb{Z}$). For the highest weight $\nu_\sigma$ of $\sigma \in \widehat{M}$ we have

$$\nu_\sigma = (\nu_1, \ldots, \nu_{n-1}, \nu_n), \tag{2.5}$$

where $\nu_1 \geq \cdots \geq \nu_{n-1} \geq |\nu_n|$ and $\nu_i$, $i = 1, \ldots, n$ are all half integers [2, p. 20]. Let $(s, S)$ be the spin representation of $K$ [8, p. 14]. Let $(s^+, S^+)$, $(s^-, S^-)$ be the half-spin representations of $M$ [8, p. 22]. The highest weight of $s$ is given by $\nu_s = (\frac{1}{2}, \ldots, \frac{1}{2})$, and the highest weights of $s^+$, $s^-$ are $\nu_{s^+} = (\frac{1}{2}, \ldots, \frac{1}{2})$, $\nu_{s^-} = (\frac{1}{2}, \ldots, -\frac{1}{2})$, respectively [2, p. 20].

### 2.3 Haar measure on *G*

The invariant measure $dg$ on $G$ can be normalized as follows. The Haar measure on $K$ is normalized such that $K$ has volume 1. Every $a \in A$ can be written as $a = \exp\log a$, where $\log a \in \mathfrak{a}$ is unique. For $t \in \mathbb{R}$, we let $a(t) := \exp(tH)$. We let

$$\langle X, Y\rangle_\theta := -\frac{1}{2(d-1)}B(X, \theta(Y)).$$

We fix an isometric identification of $\mathbb{R}^{d-1}$ with $\mathfrak{n}$ with respect to the inner product $\langle\cdot,\cdot\rangle_\theta$. The measure on $\mathfrak{n}$ is induced from the Lebesgue measure under this identification. We identify $\mathfrak{n}$ and $N$ by the exponential map. The Haar measure $dn$ on $N$ is induced from the measure on $\mathfrak{n}$ under this identification. By [30, Proposition 7.6.4], the Haar measure d$g$ on $G$ is normalized such that for $f \in C_c(G)$,

$$\int_G f(g)\mathrm{d}g = \int_K\int_{\mathbb{R}}\int_N f(ka(t)n)e^{2\rho(tH)}\mathrm{d}n\mathrm{d}t\mathrm{d}k, \tag{2.6}$$

where $\rho$ is as in (2.3).

### 2.4 The Casimir element

Let $B_0(X, Y) := \mathrm{Tr}(XY)$ be the trace form on $\mathfrak{g}\times\mathfrak{g}$. We put $C(X, Y) := B_0(X, Y)$. We choose a basis $(X_i)$ for $\mathfrak{g}$ and set $c_{ij} = C(X_i, X_j)$. Then, since $C(\cdot,\cdot)$ is a non-degenerate form, the matrix $C := (c_{ij})$ is non-singular. We denote the inverse matrix of $C$ by $C^{-1} = (c^{ij}) := (c_{ij})^{-1}$. We put $X^j = \sum c^{ij}X_i$, so that $X_i = \sum c_{ji}X^j$. Let $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ be the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. We define the Casimir element $\Omega \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ by

$$\Omega := \sum_{i,j} X^i X^j.$$

By [12, Proposition 8.6], $\Omega$ is independent of the basis $(X_i)$. Furthermore, it satisfies $\mathrm{Ad}(g)\Omega = \Omega$ for all $g \in G$ and hence is in the center $Z(\mathfrak{g}_{\mathbb{C}})$ of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$. Let $(\cdot,\cdot)$ be the inner product on $\mathfrak{g}$, defined by $(X_1, X_2) = -B_0(X_1, \theta X_2)$. Let $(X_i)$ be an orthonormal basis of $\mathfrak{p}$ and $(Y_j)$ an orthonormal basis of $\mathfrak{k}$, with respect to this inner product. By the definition of the Casimir element, we have

$$\Omega = \sum_i X_i^2 - \sum_j Y_j^2$$
$$\Omega_K = -\sum_j Y_j^2.$$

Here, $\Omega_K$ denotes the Casimir element, which corresponds to the restriction $(\cdot,\cdot)|_{\mathfrak{k}\times\mathfrak{k}}$. It lies in the center $Z(\mathfrak{k})$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{k})$ of $\mathfrak{k}$. If we consider a finite dimensional unitary irreducible representation $(\tau, V_\tau)$ of $K$ and since $\Omega \in Z(\mathfrak{k})$, Schur's Lemma [12, Proposition 1.5] implies that $\tau(\Omega_K)$ acts by a scalar $\lambda_\tau$, i.e.,

$$\tau(\Omega_K) = \lambda_\tau \operatorname{Id}_{V_\tau}.$$

The scalar $\lambda_\tau$ is called the Casimir eigenvalue of $\tau$.

### 2.5 The principal series representation

We recall the parametrization of the principal series representation. Let $P = MAN$ be the standard parabolic subgroup of $G$. For $(\sigma, V_\sigma) \in \widehat{M}$, we define the space $\mathcal{H}_{\sigma,\lambda}$ of continuous functions on $G$ by

$$\mathcal{H}_{\sigma,\lambda} := \left\{ f \in C(G, V_\sigma) \colon f(gman) = e^{-(i\lambda+|\rho|)\log a}\sigma^{-1}(m)f(g), \forall g \in G, \forall man \in P \right\},$$

where $\lambda \in \mathbb{C}$ and $\rho$ as in (2.3), with norm

$$\|f\|_c = \int_K \|f(k)\|^2 \mathrm{d}k. \tag{2.7}$$

We define the principal series representation as the induced representation

$$\pi_{\sigma,\lambda} := \operatorname{Ind}_P^G \left( \sigma \otimes e^{i\lambda} \otimes \operatorname{Id} \right),$$

with representation space the Hilbert space, obtained by completion of $\mathcal{H}_{\sigma,\lambda}$ with respect to the norm $\|\cdot\|_c$ in (2.7). For $f \in \mathcal{H}_{\sigma,\lambda}$, the action of $G$ on $f$ is given by $\pi_{\sigma,\lambda}(g)f(g') = f(g^{-1}g')$. We denote by $\mathfrak{a}_{\mathbb{C}}^*$ the space of the linear functionals on $\mathfrak{a}_{\mathbb{C}}$. In the definition of the space $\mathcal{H}_{\sigma,\lambda}$, $\lambda$ is a complex number. $\mathfrak{a}_{\mathbb{C}}^*$ is identified with $\mathbb{C}$, using the positive root. If $\lambda \in \mathbb{R}$, then the representation $\pi_{\sigma,\lambda}$ is unitary. In addition, if $\lambda \in \mathbb{R} - \{0\}$, then $\pi_{\sigma,\lambda}$ is irreducible.

### 2.6 Plancherel measure

Let $\mu_{PL}(\pi_{\sigma,\lambda})$ be the Plancherel measure, viewed as a measure on the set of the principal series representations $\pi_{\sigma,\lambda}$. Since $\operatorname{rank}(G) > \operatorname{rank}(K)$, by classical result of Harish-Chandra [11], the set of the discrete series representations of $G$ is empty. By [12, Theorem 13.2],

$$d\mu_{PL}(\pi_{\sigma,\lambda}) = P_\sigma(i\lambda)\mathrm{d}\lambda,$$

where $P_\sigma(i\lambda)$ is the Plancherel polynomial given by

$$P_\sigma(i\lambda) = \prod_{\alpha\in\Delta^+(\mathfrak{g}_{\mathbb{C}},\mathfrak{h})} \frac{\langle i\lambda + \nu_\sigma + \rho_{\mathfrak{m}}, \alpha\rangle}{\langle \rho_{\mathfrak{g}}, \alpha\rangle},$$

where $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$ is the set of the positive roots of the system $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$, $\langle \cdot, \cdot \rangle$ is defined by (2.1), $\nu_\sigma$ is the highest weight of $\sigma$ as in (2.5), $\rho_{\mathfrak{m}}$ is defined as in (2.4), and $\rho_{\mathfrak{g}}$ is defined by

$$\rho_{\mathfrak{g}} := \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})} \alpha,$$

Let $z = i\lambda \in \mathbb{C}$. By [19, pp. 264–265], $P_\sigma(z)$ is an even polynomial of $z$ and hence $P_\sigma(z) = P_\sigma(-z)$.

## 3 Twisted Selberg and Ruelle zeta functions

We consider the twisted Ruelle and Selberg zeta functions associated with the geodesic flow on the sphere vector bundle $S(X)$ of $X = \Gamma \backslash G/K$. Since $K$ acts transitively on the unit vectors in $\mathfrak{p}$, $S(\widetilde{X})$ can be represented by the homogeneous space $G/M$. Therefore, $S(X) = \Gamma \backslash G/M$.

We recall the Cartan decomposition $G = KA^+K$ of $G$, where $A^+$ is as in (2.2). Then, every element $g \in G$ can be written as $g = ha_+k$, where $h, k \in K$ and $a_+ = \exp(tH)$ for some $t \in \mathbb{R}^+$. The positive real number $t$ equals $d(eK, gK)$, where $d$ denotes the geodesic distance on $\widetilde{X}$. It is a well-known fact that there is a 1–1 correspondence between the closed geodesics on a manifold $X$ with negative sectional curvature and the non-trivial conjugacy classes of the fundamental group $\pi_1(X)$ of $X$ [10]. The hyperbolic elements of $\Gamma$ can be realized as the semisimple elements of this group, i.e., the diagonalizable elements of $\Gamma$. Since $\Gamma$ is a cocompact subgroup of $G$, every element $\gamma \in \Gamma - \{e\}$ hyperbolic. We denote by $c_\gamma$ the closed geodesic on $X$ associated with the hyperbolic conjugacy class $[\gamma]$. We denote also by $l(\gamma)$ the length of $c_\gamma$. Since $\Gamma$ is torsion-free, $l(\gamma)$ is always positive and therefore we can obtain an infimum for the length spectrum $\mathrm{spec}(\Gamma) := \{l(\gamma) : \gamma \in \Gamma\}$. An element $\gamma \in \Gamma$ is called primitive if there exists no $n \in \mathbb{N}$ with $n > 1$ and $\gamma_0 \in \Gamma$ such that $\gamma = \gamma_0^n$. A primitive element $\gamma_0 \in \Gamma$ corresponds to a geodesic on $X$. The prime geodesics correspond to the periodic orbits of minimal length. Hence, if a hyperbolic element $\gamma$ in $\Gamma$ is generated by a primitive element $\gamma_0$, then there exists a $n_\Gamma(\gamma) \in \mathbb{N}$ such that $\gamma = \gamma_0^{n_\Gamma(\gamma)}$. The corresponding closed geodesic is of length $l(\gamma) = n_\Gamma(\gamma) l(\gamma_0)$.

We lift now the closed geodesic $c_\gamma$ to the universal covering $\widetilde{X}$. For $\gamma \in \Gamma$, $l(\gamma) := \inf\{d(x, \gamma x) \colon x \in \widetilde{X}\}$, or $l(\gamma) = \inf\{d(eK, g^{-1}\gamma gK) \colon g \in G\}$. Hence, we see that the length of the closed geodesic $l(\gamma)$ depends only on $\gamma \in \Gamma$. Let $\gamma \in \Gamma$, with $\gamma \neq e$. Then, by [31, Lemma 6.5] there exist a $g \in G$, a $m_\gamma \in M$, and an $a_\gamma \in A^+$, such that $g^{-1}\gamma g = m_\gamma a_\gamma$. The element $m_\gamma$ is determined up to conjugacy in $M$, and the element $a_\gamma$ depends only on $\gamma$. Similar to the consideration of [2, Section 3.1], the geodesic flow $\phi$ on $S(X)$ is given by the map $\phi \colon \mathbb{R} \times S(X) \ni (t, \Gamma gM) \to \Gamma g \exp(-tH)M \in S(X)$. A closed orbit of $\phi$ is described by the set $c := \{\Gamma g \exp(-tH)M \colon t \in \mathbb{R}\}$, where $g \in G$ is such that $g^{-1}\gamma g := m_\gamma a_\gamma \in MA^+$. The Anosov property of the geodesic flow $\phi$ on $S(X)$ can be expressed by the following $d\phi$-invariant splitting of $TS(X)$

$$TS(X) = T^s S(X) \oplus T^c S(X) \oplus T^u S(X), \tag{3.1}$$

where $T^s S(X)$ consist of vectors that shrink exponentially, $T^u S(X)$ expand exponentially, and $T^c S(X)$ is the one-dimensional subspace of vectors tangent to the flow, with respect to the Riemannian metric, as $t \to \infty$. The spitting in (3.1) corresponds to splitting

$$TS(X) = \Gamma \backslash G \times_{\mathrm{Ad}} (\overline{\mathfrak{n}} \oplus \mathfrak{a} \oplus \mathfrak{n}), \tag{3.2}$$

where Ad denotes the adjoint action of $\mathrm{Ad}(\exp(-tH))$ on $\overline{\mathfrak{n}}$, $\mathfrak{a}$, $\mathfrak{n}$, and $\overline{\mathfrak{n}} = \theta\mathfrak{n}$ is the sum of the negative root spaces of the system $(\mathfrak{g}, \mathfrak{a})$.

**Definition 3.1** Let $\chi : \Gamma \to \mathrm{GL}(V_\chi)$ be a finite dimensional representation of $\Gamma$ and $\sigma \in \widehat{M}$. The twisted Selberg zeta function $Z(s; \sigma, \chi)$ for $X$ is defined by the infinite product

$$Z(s;\sigma,\chi) := \prod_{\substack{[\gamma]\neq e \\ [\gamma]\,\text{prime}}} \prod_{k=0}^{\infty} \det\left(\mathrm{Id} - (\chi(\gamma)\otimes\sigma(m_\gamma)\otimes S^k(\mathrm{Ad}(m_\gamma a_\gamma)|_{\overline{\mathfrak{n}}}))e^{-(s+|\rho|)l(\gamma)}\right), \tag{3.3}$$

where $s \in \mathbb{C}$, $S^k(\mathrm{Ad}(m_\gamma a_\gamma)_{\overline{\mathfrak{n}}})$ denotes the $k$th symmetric power of the adjoint map $\mathrm{Ad}(m_\gamma a_\gamma)$ restricted to $\overline{\mathfrak{n}}$ and $\rho$ is as in (2.3).

**Definition 3.2** Let $\chi : \Gamma \to \mathrm{GL}(V_\chi)$ be a finite dimensional representation of $\Gamma$ and $\sigma \in \widehat{M}$. The twisted Ruelle zeta function $R(s; \sigma, \chi)$ for $X$ is defined by the infinite product

$$R(s;\sigma,\chi) := \prod_{\substack{[\gamma]\neq e \\ [\gamma]\,\text{prime}}} \det\left(\mathrm{Id} - (\chi(\gamma)\otimes\sigma(m_\gamma))e^{-sl(\gamma)}\right)^{(-1)^{d-1}}, \tag{3.4}$$

where $s \in \mathbb{C}$.

In order to prove the convergence of the zeta functions, we need to find an upper bound for the character of any finite dimensional representation of $\Gamma$.

**Lemma 3.3** *Let $\chi : \Gamma \to \mathrm{GL}(V_\chi)$ be a finite dimensional representation of $\Gamma$. Then, there exist positive constants $\alpha, \beta > 0$ such that*

$$|\mathrm{tr}(\chi(\gamma))| \leq \alpha e^{\beta l(\gamma)}, \quad \forall \gamma \in \Gamma - \{e\}. \tag{3.5}$$

*Proof* We fix a finite set of generators $L = \{\gamma_1, \ldots, \gamma_r\}$ of $\Gamma$ and choose a norm $\|\cdot\|$ on $V_\chi$. Let $d_W(\cdot,\cdot)$ be the word metric on $\Gamma$ (see [15]). Let $l_W(\gamma) = d_W(\gamma, e)$ be the length of $\gamma \in \Gamma$ with respect to this metric. Then, if we put $C = \max\{\|\chi(\gamma_i)\| : \gamma_i \in L \cup L^{-1}\}$, we get for $c = \log C$,

$$\|\chi(\gamma)\| \leq e^{c l_W(\gamma)}. \tag{3.6}$$

By [15, Proposition 3.2], it follows that there exist positive constants $c_1, c_2 > 0$ such that

$$c_1 d(x_0, \gamma x_0) \leq d_W(e, \gamma) \leq c_2 d(x_0, \gamma x_0), \tag{3.7}$$

where $x_0 := eK$. Then, (3.6) becomes by (3.7)

$$\|\chi(\gamma)\| \leq C_1 e^{c_2 d(x_0, \gamma x_0)}.$$

It follows that

$$|\mathrm{tr}\,\chi(\gamma)| \leq \dim(V_\chi)\|\chi(\gamma)\| \leq C_3 e^{c_2 d(x_0,\gamma x_0)}. \tag{3.8}$$

Recall that

$$l(\gamma) := \min\left\{d(x, \gamma x) : x \in \widetilde{X}\right\}.$$

We choose a fundamental domain $F \subset \widetilde{X}$ for $\Gamma$ such that $x_0 \in F$. Given $\gamma \in \Gamma$, let $x_1 \in \widetilde{X}$, such that $l(\gamma) = d(x_1, \gamma x_1)$. Then, there exists a $\gamma_1 \in \Gamma$ such that $x_1 \in \gamma_1 F$. Let $x_2 \in F$

such that $x_1 = \gamma_1 x_2$. By compactness of the fundamental domain, $\mathrm{diam}(F)$ is finite. If we put $\delta := \mathrm{diam}(F)$, then

$$d(x_0, x_2) \leq \delta. \tag{3.9}$$

We see that

$$\begin{aligned} d(x_0, \gamma_1^{-1}\gamma\gamma_1 x_0) &\leq d(x_0, x_2) + d(x_2, \gamma_1^{-1}\gamma\gamma_1 x_0) \\ &\leq \delta + d(x_2, \gamma_1^{-1}\gamma\gamma_1 x_0). \end{aligned} \tag{3.10}$$

In addition,

$$\begin{aligned} d\left(x_2, \gamma_1^{-1}\gamma\gamma_1 x_0\right) &\leq d\left(x_2, \gamma_1^{-1}\gamma\gamma_1 x_2\right) + d\left(\gamma_1^{-1}\gamma\gamma_1 x_2, \gamma_1^{-1}\gamma\gamma_1 x_0\right) \\ &\leq d\left(x_2, \gamma_1^{-1}\gamma\gamma_1 x_2\right) + d(x_0, x_2) \\ &\leq d\left(x_2, \gamma_1^{-1}\gamma\gamma_1 x_2\right) + \delta. \end{aligned} \tag{3.11}$$

Hence, by (3.10) and (3.11) we get

$$d\left(x_0, \gamma_1^{-1}\gamma\gamma_1 x_0\right) \leq 2\delta + d\left(x_2, \gamma_1^{-1}\gamma\gamma_1 x_2\right).$$

Recall that $x_1 = \gamma_1 x_2$. Therefore, we have

$$\begin{aligned} d\left(x_0, \gamma_1^{-1}\gamma\gamma_1 x_0\right) &\leq 2\delta + d\left(\gamma_1^{-1} x_1, \gamma_1^{-1}\gamma x_1\right) \\ &\leq 2\delta + d(x_1, \gamma x_1). \end{aligned} \tag{3.12}$$

Using (3.8) and (3.12) we obtain the following inequalities.

$$\begin{aligned} |\mathrm{tr}(\chi(\gamma))| &= |\mathrm{tr}(\chi(\gamma_1^{-1}\gamma\gamma_1))| \\ &\leq C_3 e^{c_2 d(x_0, \gamma_1^{-1}\gamma\gamma_1 x_0)} \\ &\leq C_3 e^{c_2(2\delta + d(x_1, \gamma x_1))} \\ &= C_4 e^{c_2 d(x_1, \gamma x_1)} = C_4 e^{c_2 l(\gamma)}. \end{aligned}$$

The assertion follows. □

**Proposition 3.4** *Let* $\chi\colon \Gamma \to \mathrm{GL}(V_\chi)$ *be a finite dimensional representation of* $\Gamma$ *and* $\sigma \in \widehat{M}$. *Then, there exists a constant* $c > 0$ *such that*

$$Z(s; \sigma, \chi) := \prod_{\substack{[\gamma]\neq e \\ [\gamma]\,\mathrm{prime}}} \prod_{k=0}^{\infty} \det\left(\mathrm{Id} - (\chi(\gamma) \otimes \sigma(m_\gamma) \otimes S^k(\mathrm{Ad}(m_\gamma a_\gamma)_{\overline{\mathfrak{n}}}))e^{-(s+|\rho|)l(\gamma)}\right), \tag{3.13}$$

*converges absolutely and uniformly on compact subsets of the half-plane* $\mathrm{Re}(s) > c$.

*Proof* We observe that

$$\begin{aligned}\log Z(s;\sigma,\chi) &= \sum_{\substack{[\gamma]\neq e\\ [\gamma]\,\text{prime}}}\sum_{k=0}^{\infty}\operatorname{tr}\log\Big(1-(\chi(\gamma)\otimes\sigma(m_\gamma)\otimes S^k(\operatorname{Ad}(m_\gamma a_\gamma)_{\overline{\mathfrak{n}}}))e^{-(s+|\rho|)l(\gamma)}\Big)\\
&= -\sum_{\substack{[\gamma]\neq e\\ [\gamma]\,\text{prime}}}\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\frac{\operatorname{tr}\big((\chi(\gamma)\otimes\sigma(m_\gamma)\otimes S^k(\operatorname{Ad}(m_\gamma a_\gamma)_{\overline{\mathfrak{n}}}))e^{-(s+|\rho|)l(\gamma)}\big)^j}{j}\\
&= -\sum_{[\gamma]\neq e}\sum_{k=0}^{\infty}\frac{1}{n_\Gamma(\gamma)}\operatorname{tr}\Big(\chi(\gamma)\otimes\sigma(m_\gamma)\otimes S^k\left(\operatorname{Ad}(m_\gamma a_\gamma)_{\overline{\mathfrak{n}}}\right)\Big)e^{-(s+|\rho|)l(\gamma)}\\
&= -\sum_{[\gamma]\neq e}\frac{1}{n_\Gamma(\gamma)}\operatorname{tr}(\chi(\gamma)\otimes\sigma(m_\gamma))\frac{e^{-(s+|\rho|)l(\gamma)}}{\det(\operatorname{Id}-\operatorname{Ad}(m_\gamma a_\gamma)_{\overline{\mathfrak{n}}})},\end{aligned}\tag{3.14}$$

where in the last equation we made use of the identity

$$\sum_{k=0}^{\infty}\operatorname{tr}\,S^k(\operatorname{Ad}(m_\gamma a_\gamma)_{\overline{\mathfrak{n}}})=\frac{1}{\det(\operatorname{Id}-\operatorname{Ad}(m_\gamma a_\gamma)_{\overline{\mathfrak{n}}})},$$

since $\operatorname{Ad}(m_\gamma a_\gamma)_{\overline{\mathfrak{n}}}$ represents an endomorphism of $\overline{\mathfrak{n}}$ with eigenvalues of modulus less than one. In particular, we have $\det(\operatorname{Ad}(a_\gamma)_{\overline{\mathfrak{n}}})=\exp(-2|\rho|l(\gamma))$. We observe that

$$|\operatorname{tr}\sigma(m_\gamma)|\leq\dim(\sigma),\quad\forall\sigma\in\widehat{M}.$$

Using the normalization of the Haar measure on $G$ as in Sect. 2, we see that there exists a positive constant $C>0$ such that for every $R>0$

$$\operatorname{Vol}(B(x_0,R))\leq Ce^{2|\rho|R},$$

where $\rho$ is as in (2.3). Since $\Gamma$ is a cocompact lattice of $G$, there exists a positive constant $C'$ such that

$$\sharp\{[\gamma]:l(\gamma)<R\}\leq\sharp\{\gamma\in\Gamma:l(\gamma)\leq R\}\leq C'e^{2|\rho|R}\tag{3.15}$$

[2, (1.31)]. We need an upper bound for the quantity

$$\frac{1}{\det\left(\operatorname{Id}-\operatorname{Ad}(m_\gamma a_\gamma)_{\overline{\mathfrak{n}}}\right)}.$$

We use the estimates (3.15) to see that we can consider a $[\gamma_{min}]$ among all the conjugacy classes of $\Gamma$, such that $l(\gamma_{min})$ is of minimum length. Hence, there exists a positive constant $C''>0$ such that

$$\frac{1}{\det\left(\operatorname{Id}-\operatorname{Ad}(m_\gamma a_\gamma)_{\overline{\mathfrak{n}}}\right)}<C''.$$

Let

$$\mathcal{N}(R):=\sharp\{[\gamma]\in\Gamma:l(\gamma)\leq R\},\quad R\geq 0.$$

By Lemma 3.3, it follows that there exist positive constants $C, c_1 > 0$ such that

$$\begin{aligned}\sum_{[\gamma]\neq e}\frac{1}{n_\Gamma(\gamma)}\Big|\operatorname{tr}(\chi(\gamma)\otimes\sigma(m_\gamma))\frac{e^{-(s+|\rho|)l(\gamma)}}{\det(\operatorname{Id}-\operatorname{Ad}(m_\gamma a_\gamma)_{\overline{\mathfrak{n}}})}\Big| &\leq C\sum_{[\gamma]\neq e}e^{(c_1-\operatorname{Re}(s))l(\gamma)}\\ &= C\sum_{k=0}^{\infty}\sum_{\substack{[\gamma]\neq e\\ k\leq l(\gamma)\leq k+1}}e^{(c_1-\operatorname{Re}(s))l(\gamma)}\\ &\leq C\sum_{k=0}^{\infty}\mathcal{N}(k+1)e^{(c_1-\operatorname{Re}(s))k}.\end{aligned}$$

By (3.15), we have

$$\sum_{k=0}^{\infty}\mathcal{N}(k+1)e^{(c_1-\operatorname{Re}(s))k}\leq C'\sum_{k=0}^{\infty}e^{(2|\rho|+c_1-\operatorname{Re}(s))k}.$$

Hence, there exists a positive constant $c > 0$ such that for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > c$,

$$\sum_{[\gamma]\neq e}\frac{1}{n_\Gamma(\gamma)}\Big|\operatorname{tr}(\chi(\gamma)\otimes\sigma(m_\gamma))\frac{e^{-(s+|\rho|)l(\gamma)}}{\det(\operatorname{Id}-\operatorname{Ad}(m_\gamma a_\gamma)_{\overline{\mathfrak{n}}})}\Big|<\infty.$$

The assertion follows from (3.14). □

**Proposition 3.5** *Let $\chi\colon \Gamma \to \operatorname{GL}(V_\chi)$ be a finite dimensional representation of $\Gamma$ and $\sigma \in \widehat{M}$. Then, there exists a constant $r > 0$ such that*

$$R(s;\sigma,\chi) := \prod_{\substack{[\gamma]\neq e\\ [\gamma]\text{ prime}}}\det\left(\operatorname{Id}-(\chi(\gamma)\otimes\sigma(m_\gamma))e^{-sl(\gamma)}\right)^{(-1)^{d-1}}. \tag{3.16}$$

*converges absolutely and uniformly on compact subsets of the half-plane* $\operatorname{Re}(s) > r$.

*Proof* We observe that

$$\begin{aligned}\log R(s;\sigma,\chi) &= (-1)^{d-1}\sum_{\substack{[\gamma]\neq e\\ [\gamma]\text{ prime}}}\operatorname{tr}\log\left(1-(\chi(\gamma)\otimes\sigma(m_\gamma))e^{-sl(\gamma)}\right)\\ &= (-1)^{d}\sum_{\substack{[\gamma]\neq e\\ [\gamma]\text{ prime}}}\sum_{j=1}^{\infty}\frac{\operatorname{tr}\left(((\chi(\gamma)\otimes\sigma(m_\gamma))e^{-sl(\gamma)})^j\right)}{j}\\ &= (-1)^{d}\sum_{[\gamma]\neq e}\frac{1}{n_\Gamma(\gamma)}\operatorname{tr}(\chi(\gamma)\otimes\sigma(m_\gamma))e^{-sl(\gamma)}.\end{aligned} \tag{3.17}$$

By Lemma 3.3, it follows that there exist positive constants $C, c_1 > 0$ such that

$$\begin{aligned}\sum_{[\gamma]\neq e} \frac{1}{n_\Gamma(\gamma)}\left| \operatorname{tr}(\chi(\gamma)\otimes\sigma(m_\gamma))e^{-sl(\gamma)}\right| &\leq C \sum_{[\gamma]\neq e} e^{(c_1-\operatorname{Re}(s))l(\gamma)} \\ &= C\sum_{k=0}^{\infty} \sum_{\substack{[\gamma]\neq e\\ k\leq l(\gamma)\leq k+1}} \mathcal{N}(k+1)e^{(c_1-\operatorname{Re}(s))l(\gamma)} \\ &\leq C\sum_{k=0}^{\infty}\mathcal{N}(k+1)e^{(c_1-\operatorname{Re}(s))k}.\end{aligned}$$

By (3.15), we have

$$\sum_{k=0}^{\infty}\mathcal{N}(k+1)e^{(c_1-\operatorname{Re}(s))k} \leq C'\sum_{k=0}^{\infty} e^{(2|\rho|+c_1-\operatorname{Re}(s))k}.$$

Hence, there exists a positive constant $r > 0$ such that for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > r$,

$$\sum_{[\gamma]\neq e} \frac{1}{n_\Gamma(\gamma)}\left| \operatorname{tr}(\chi(\gamma)\otimes\sigma(m_\gamma))e^{-sl(\gamma)}\right| < \infty. \tag{3.18}$$

The assertion follows from (3.17). □

Let $w$ be the non-trivial element of the restricted Weyl group $W_A$. For a non-Weyl invariant representation $\sigma \in \widehat{M}$, we define the following twisted zeta functions.

**Definition 3.6** The symmetrized zeta function $S(s; \sigma, \chi)$ for $X$ is defined by

$$S(s;\sigma,\chi) := Z(s;\sigma,\chi)Z(s;w\sigma,\chi). \tag{3.19}$$

**Definition 3.7** The super zeta function $Z^s(s; \sigma, \chi)$ for $X$ is defined by

$$Z^s(s;\sigma,\chi) := \frac{Z(s;\sigma,\chi)}{Z(s;w\sigma,\chi)}. \tag{3.20}$$

**Definition 3.8** The super Ruelle zeta function $R^s(s; \sigma, \chi)$ for $X$ is defined by

$$R^s(s;\sigma,\chi) := \frac{R(s;\sigma,\chi)}{R(s;w\sigma,\chi)}. \tag{3.21}$$

**Lemma 3.9** *Let*

$$L(\gamma;\sigma,\chi) := \frac{\operatorname{tr}(\chi(\gamma)\otimes\sigma(m_\gamma))e^{-|\rho|l(\gamma)}}{\det(\operatorname{Id}-\operatorname{Ad}(m_\gamma a_\gamma)_{\overline{n}})}. \tag{3.22}$$

*Then,*

1. *the logarithmic derivative of the Selberg zeta function $Z(s; \sigma, \chi)$ is given by*

$$L(s) := \frac{d}{ds}\log(Z(s;\sigma,\chi)) = \sum_{[\gamma]\neq e}\frac{l(\gamma)}{n_\Gamma(\gamma)}L(\gamma;\sigma,\chi)e^{-sl(\gamma)}. \tag{3.23}$$

2. *the logarithmic derivative of the symmetrized zeta function* $S(s;\sigma,\chi)$ *is given by*

$$L_S(s) := \frac{\mathrm{d}}{\mathrm{d}s}\log(S(s;\sigma,\chi)) = \sum_{[\gamma]\neq e}\frac{l(\gamma)}{n_\Gamma(\gamma)}L(\gamma;\sigma+w\sigma,\chi)e^{-sl(\gamma)}. \tag{3.24}$$

3. *the logarithmic derivative of the super zeta function* $Z^s(s;\sigma,\chi)$ *is given by*

$$L^s(s) := \frac{\mathrm{d}}{\mathrm{d}s}\log(Z^s(s;\sigma,\chi)) = \sum_{[\gamma]\neq e}\frac{l(\gamma)}{n_\Gamma(\gamma)}L(\gamma;\sigma-w\sigma,\chi)e^{-sl(\gamma)}. \tag{3.25}$$

*Proof* 1. We see by Eq. (3.14)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\log(Z(s;\sigma,\chi)) &= \sum_{[\gamma]\neq e}\frac{l(\gamma)}{n_\Gamma(\gamma)}\operatorname{tr}(\chi(\gamma)\otimes\sigma(m_\gamma))\frac{e^{-sl(\gamma)}e^{-|\rho|l(\gamma)}}{\det(\mathrm{Id}-\mathrm{Ad}(m_\gamma a_\gamma)_{\overline{\mathfrak{n}}})}\\ &= \sum_{[\gamma]\neq e}\frac{l(\gamma)}{n_\Gamma(\gamma)}L(\gamma;\sigma,\chi)e^{-sl(\gamma)}.\end{aligned}$$

2. For the symmetrized zeta function $S(s;\sigma,\chi)$, we see by Eq. (3.19)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\log(S(s;\sigma,\chi)) &= \frac{\mathrm{d}}{\mathrm{d}s}\log(Z(s;\sigma,\chi) + \frac{\mathrm{d}}{\mathrm{d}s}\log(Z(s;w\sigma,\chi)\\ &= \sum_{[\gamma]\neq e}\frac{l(\gamma)}{n_\Gamma(\gamma)}\operatorname{tr}(\chi(\gamma)\otimes\sigma(m_\gamma))\frac{e^{-sl(\gamma)}e^{-|\rho|l(\gamma)}}{\det(\mathrm{Id}-\mathrm{Ad}(m_\gamma a_\gamma)_{\overline{\mathfrak{n}}})}\\ &\quad+\sum_{[\gamma]\neq e}\frac{l(\gamma)}{n_\Gamma(\gamma)}\operatorname{tr}(\chi(\gamma)\otimes w\sigma(m_\gamma))\frac{e^{-sl(\gamma)}e^{-|\rho|l(\gamma)}}{\det(\mathrm{Id}-\mathrm{Ad}(m_\gamma a_\gamma)_{\overline{\mathfrak{n}}})}\\ &= \sum_{[\gamma]\neq e}\frac{l(\gamma)}{n_\Gamma(\gamma)}L(\gamma;\sigma+w\sigma,\chi)e^{-sl(\gamma)}.\end{aligned}$$

3. For the super zeta function $Z^s(s;\sigma,\chi)$, we see by Eq. (3.20)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\log(Z^s(s;\sigma,\chi)) &= \frac{\mathrm{d}}{\mathrm{d}s}\log(Z(s;\sigma,\chi) - \frac{\mathrm{d}}{\mathrm{d}s}\log(Z(s;w\sigma,\chi)\\ &= \sum_{[\gamma]\neq e}\frac{l(\gamma)}{n_\Gamma(\gamma)}\operatorname{tr}(\chi(\gamma)\otimes\sigma(m_\gamma))\frac{e^{-sl(\gamma)}e^{|\rho|l(\gamma)}}{\det\left(\mathrm{Id}-\mathrm{Ad}(m_\gamma a_\gamma)_{\overline{\mathfrak{n}}}\right)}\\ &\quad-\sum_{[\gamma]\neq e}\frac{l(\gamma)}{n_\Gamma(\gamma)}\operatorname{tr}(\chi(\gamma)\otimes w\sigma(m_\gamma))\frac{e^{-sl(\gamma)}e^{|\rho|l(\gamma)}}{\det\left(\mathrm{Id}-\mathrm{Ad}(m_\gamma a_\gamma)_{\overline{\mathfrak{n}}}\right)}\\ &= \sum_{[\gamma]\neq e}\frac{l(\gamma)}{n_\Gamma(\gamma)}L(\gamma;\sigma-w\sigma,\chi)e^{-sl(\gamma)}.\end{aligned}$$

□

## 4 The twisted Bochner–Laplace operator

Let $E\to X$ be a complex vector bundle with covariant derivative $\nabla$. We define the second covariant derivative $\nabla^2$ by

$$\nabla^2_{V,W} := \nabla_V\nabla_W - \nabla_{\nabla^{LC}_V W},$$

where $V, W \in C^\infty(X, TX)$ and $\nabla^{LC}$ denotes the Levi-Civita connection on $TX$. We define the connection Laplacian $\Delta_E$ to be the negative of the trace of the second covariant derivative, i.e.,

$$\Delta_E := -\operatorname{Tr} \nabla^2.$$

If $E$ is equipped with a Hermitian metric compatible with the connection $\nabla$, then by [14, p. 154] the connection Laplacian is equal to the Bochner–Laplace operator, i.e.,

$$\Delta_E = \nabla^* \nabla.$$

In terms of a local orthonormal frame field $(e_1, \ldots, e_d)$ of $T_x X$, for $x \in X$, the connection Laplacian is given by

$$\Delta_E = -\sum_{j=1}^{d} \nabla^2{}_{e_j, e_j}.$$

The principal symbol of $\Delta_E$ equals

$$\sigma_{\Delta_E}(x, \xi) = \|\xi\|^2 \operatorname{Id}_{E_x},$$

where $x \in X$, $\xi \in \mathrm{T}_x^* X$. $\Delta_E$ acts in $L^2(X, E)$ with domain $C^\infty(X, E)$. $\Delta_E \colon C^\infty(X, E) \circlearrowright$ is a second-order elliptic formally self-adjoint differential operator.

Let $\chi \colon \Gamma \to \mathrm{GL}(V_\chi)$ be a finite dimensional representation of $\Gamma$. Let $E_\chi \to X$ be the associated flat vector bundle over $X$, equipped with a flat connection $\nabla^{E_\chi}$. We specialize to the twisted case $E = E_0 \otimes E_\chi$, where $E_0 \to X$ is a complex vector bundle equipped with a connection $\nabla^{E_0}$ and a metric, which is compatible with this connection. Let $\nabla^E = \nabla^{E_0 \otimes E_\chi}$ be the product connection, defined by

$$\nabla^{E_0 \otimes E_\chi} := \nabla^{E_0} \otimes 1 + 1 \otimes \nabla^{E_\chi}.$$

We define the operator $\Delta^\sharp_{E_0, \chi}$ by

$$\Delta^\sharp_{E_0, \chi} := -\operatorname{Tr}\left((\nabla^{E_0 \otimes E_\chi})^2\right). \tag{4.1}$$

We choose a Hermitian metric in $E_\chi$. Then, $\Delta^\sharp_{E_0, \chi}$ acts on $L^2(X, E_0 \otimes E_\chi)$. However, it is not a formally self-adjoint operator in general. We want to describe this operator locally. Following the analysis in [23], we consider an open subset $U$ of $X$ such that $E_\chi|_U$ is trivial. Then, $E_0 \otimes E_\chi|_U$ is isomorphic to the direct sum of $m$-copies of $E_0|_U$, i.e.,

$$(E_0 \otimes E_\chi)|_U \cong \oplus_{i=1}^m E_0|_U,$$

where $m := \operatorname{rank}(E_\chi) = \dim V_\chi$. Let $(e_i), i = 1, \ldots, m$ be any basis of flat sections of $E_\chi|_U$. Then, each $\phi \in C^\infty(U, (E_0 \otimes E_\chi)|_U)$ can be written as

$$\phi = \sum_{i=1}^m \phi_i \otimes e_i,$$

where $\phi_i \in C^\infty(U, E_0|_U)$, $i = 1, \ldots, m$. The product connection is given by

$$\nabla_Y^{E_0 \otimes E_\chi} \phi = \sum_{i=1}^m \left(\nabla_Y^{E_0}\right)(\phi_i) \otimes e_i,$$

where $Y \in C^{\infty}(X, TX)$. By (4.1) we obtain the twisted Bochner–Laplace operator acting on $C^{\infty}(X, E_0 \otimes E_{\chi})$ given by

$$\Delta^{\sharp}_{E_0,\chi}\phi = \sum_{i=1}^{m} \left(\Delta_{E_0}\phi_i\right) \otimes e_i, \tag{4.2}$$

where $\Delta_{E_0}$ denotes the Bochner–Laplace operator $\Delta^{E_0} = (\nabla^{E_0})^{*}\nabla^{E_0}$ associated with the connection $\nabla^{E_0}$. The local definition of the twisted Bochner–Laplace operator is independent of the choice of the base of flat sections of $E_{\chi}|_U$, since the transition maps comparing flat sections are constant. Let now $\widetilde{E}_0$, $\widetilde{E}_{\chi}$ be the pullbacks to $\widetilde{X}$ of $E_0$, $E_{\chi}$, respectively. Then,

$$\widetilde{E}_{\chi} \cong \widetilde{X} \times V_{\chi},$$

and

$$C^{\infty}(\widetilde{X}, \widetilde{E}_0 \otimes \widetilde{E}_{\chi}) \cong C^{\infty}(\widetilde{X}, \widetilde{E}_0) \otimes V_{\chi}. \tag{4.3}$$

With respect to the isomorphism (4.3), it follows from (4.2) that the lift of $\Delta^{\sharp}_{E_0,\chi}$ to $\widetilde{X}$ takes the form

$$\widetilde{\Delta}^{\sharp}_{E_0,\chi} = \widetilde{\Delta}_{E_0} \otimes \mathrm{Id}_{V_{\chi}}, \tag{4.4}$$

where $\widetilde{\Delta}_{E_0}$ is the lift of $\Delta_{E_0}$ to $\widetilde{X}$. By (4.2), $\Delta^{\sharp}_{E_0,\chi}$ has principal symbol

$$\sigma_{\Delta^{\sharp}_{E_0,\chi}}(x, \xi) = \|\xi\|^2_x \, \mathrm{Id}_{(E_0 \otimes E_{\chi})_x}, \quad x \in X, \xi \in \mathrm{T}^*_x X.$$

Hence, it has nice spectral properties, i.e., its spectrum is discrete and contained in a translate of a positive cone $C \subset \mathbb{C}$ such that $\mathbb{R}^+ \subset C$ (Lemma 8.6, "Appendix").

## 5 Homogeneous vector bundles and trace formulas

We specialize the twisted Bochner–Laplace operator $\Delta^{\sharp}_{E_0,\chi}$ to the case of the operator $\Delta^{\sharp}_{\tau,\chi}$ acting on smooth sections of the twisted vector bundle $E_{\tau} \otimes E_{\chi}$. Here, $E_{\tau}$ is the locally homogeneous vector bundle associated with a finite dimensional unitary representation $(\tau, V_{\tau})$ of $K$. The exact definition of the locally homogeneous vector bundle is given later in this section. The keypoint is that when we consider the lift of the twisted Bochner–Laplace operator to the universal covering, it acts as the identity operator on $V_{\chi}$. Recall that by (4.4), we have

$$\widetilde{\Delta}^{\sharp}_{\tau,\chi} = \widetilde{\Delta}_{\tau} \otimes \mathrm{Id}_{V_{\chi}},$$

where $\widetilde{\Delta}_{\tau}$ is the lift to $\widetilde{X}$ of the Bochner–Laplace operator $\Delta_{\tau}$, associated with the representation $\tau$ of $K$. We give here an explicit description of the operator $\Delta_{\tau}$. We regard the Lie group $G$ as principal $K$-fiber bundle over $\widetilde{X}$. Let $\pi : G \to G/K$ be the canonical projection. Then, since $\mathfrak{p}$ is invariant under the adjoint action $\mathrm{Ad}(k)$, for $k \in K$, the assignment

$$T^{hor}_g := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g\exp(tX), \quad X \in \mathfrak{p}$$

defines a horizontal distribution on $G$ [13, Chapter III]. This is the canonical connection on the principal bundle $G$. Let $\tau : K \to \mathrm{GL}(V_{\tau})$ be a complex finite dimensional unitary representation of $K$ on a vector space $V_\tau$, equipped with an inner product $\langle\cdot,\cdot\rangle_\tau$. Let $\widetilde{E}_\tau$ be the homogeneous vector bundle associated with $(\tau, V_\tau)$, defined by

$$\widetilde{E}_\tau := G \times_\tau V_\tau \to \widetilde{X},$$

where $K$ acts on $(G, V_\tau)$ on the right by

$$(g, v)k = (gk, \tau^{-1}(k)v), \quad g \in G, k \in K, v \in V_\tau.$$

The inner product $\langle\cdot,\cdot\rangle_\tau$ on the vector space $V_\tau$ induces a $G$-invariant metric $h^{E_\tau}$ on $\widetilde{E}_\tau$. We denote by $C^\infty(\widetilde{X}, \widetilde{E}_\tau)$ the space of the smooth sections of the vector bundle $\widetilde{E}_\tau$. We define the space

$$C^\infty(G;\tau) = \{f : G \to V_\tau : f \in C^\infty, f(gk) = \tau(k)^{-1} f(g), \forall g \in G, \forall k \in K\}. \quad (5.1)$$

Similarly, we denote by $C_c^\infty(G;\tau)$ the subspace of $C^\infty(G;\tau)$ of compactly supported functions and $L^2(G;\tau)$ the completion of $C_c^\infty(G;\tau)$ with respect to the inner product

$$\langle f, h\rangle = \int_{G/K} \langle f(g), h(g)\rangle_\tau \mathrm{d}\dot{g}.$$

Let $A\colon C^\infty(\widetilde{X}, \widetilde{E}_\tau) \to C^\infty(G;\tau)$ be the operator, defined by

$$Af(g) = g^{-1} f(gK).$$

The canonical connection on $\widetilde{E}_\tau$ is given by

$$\begin{aligned} A(\nabla^\tau_{d\pi(g)X} f)(g) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} Af(g\exp(tX)) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (g\exp(tX))^{-1} f(g\exp(tX)K), \end{aligned}$$

where $g \in G$, $X \in \mathfrak{p}$, and $f \in C^\infty(\widetilde{X}, \widetilde{E}_\tau)$. By [20, p. 4], $A$ induces a canonical isomorphism

$$C^\infty(\widetilde{X}, \widetilde{E}_\tau) \cong C^\infty(G;\tau). \quad (5.2)$$

Similarly, there exist the following isomorphisms

$$\begin{aligned} C_c^\infty(\widetilde{X}, \widetilde{E}_\tau) &\cong C_c^\infty(G;\tau); \\ L^2(\widetilde{X}, \widetilde{E}_\tau) &\cong L^2(G;\tau). \end{aligned} \quad (5.3)$$

We consider the Bochner–Laplace operator associated with $\widetilde{\nabla}^\tau$

$$\widetilde{\Delta}_\tau = (\widetilde{\nabla}^\tau)^* \widetilde{\nabla}^\tau : C_c^\infty(\widetilde{X}, \widetilde{E}_\tau) \to L^2(\widetilde{X}, \widetilde{E}_\tau).$$

Let now $\Omega \in Z(\mathfrak{g}_\mathbb{C})$ be the Casimir element of $G$. We assume that $\tau$ is irreducible. Let $\Omega|_K \in Z(\mathfrak{k})$ be the Casimir element of $K$ and $\lambda_\tau$ the associated Casimir eigenvalue. With respect to the isomorphism (5.2), the Bochner–Laplace operator acting on $C_c^\infty(G;\tau)$ is given by

$$\widetilde{\Delta}_\tau = -R(\Omega) + \lambda_\tau \operatorname{Id}. \quad (5.4)$$

This is proved in [20, Proposition 1.1]. The operator $\widetilde{\Delta}_\tau$ is an elliptic formally self-adjoint differential operator of second order. By [4], it is an essentially self-adjoint operator. Its self-adjoint extension will be also denoted by $\widetilde{\Delta}_\tau$. We consider the heat operator $e^{-t\widetilde{\Delta}_\tau}$ acting on the space $L^2(\widetilde{X}, \widetilde{E}_\tau)$

$$e^{-t\widetilde{\Delta}_\tau} \colon L^2(\widetilde{X}, \widetilde{E}_\tau) \to L^2(\widetilde{X}, \widetilde{E}_\tau).$$

By the isomorphism $L^2(\widetilde{X}, \widetilde{E}_\tau) \cong L^2(G;\tau)$ in (5.3) we can consider the heat operator as an operator in $L^2(G;\tau)$. By [3, p. 467], $e^{-t\widetilde{\Delta}_\tau}$, $t>0$ is an infinitely smoothing operator with a $C^\infty$-kernel, i.e., there exists a smooth function $k_t^\tau : G \times G \to \mathrm{End}(V_\tau)$ such that

1. it is symmetric in the $G$-variables and for each $g \in G$, the map $g' \mapsto k_t^\tau(g,g')$ belongs to $L^2(G, \mathrm{End}(V_\tau))$;
2. it satisfies the covariance property

$$k_t^\tau(gk, g'k') = \tau^{-1}(k) k_t^\tau(g,g')\tau(k'), \quad \forall g, g' \in G, k, k' \in K;$$

3. for $f \in L^2(G;\tau)$,

$$e^{-t\widetilde{\Delta}_\tau} f(g) = \int_G k_t^\tau(g,g') f(g') \mathrm{d}g'. \tag{5.5}$$

$\widetilde{\Delta}_\tau$ is a $G$-invariant operator and $e^{-t\widetilde{\Delta}_\tau}$ is an integral operator which commutes with the right regular representation of $G$ in $L^2(\widetilde{X}, \widetilde{E}_\tau)$. Then, there exists a function $H_t^\tau : G \to \mathrm{End}(V_\tau)$, such that

1. $H_t^\tau(g^{-1}g') = k_t^\tau(g,g'), \quad \forall g, g' \in G;$
2. it satisfies the covariance property

$$H_t^\tau(kgk') = \tau^{-1}(k) H_t^\tau(g)\tau(k'), \quad \forall g \in G, \forall k, k' \in K; \tag{5.6}$$

3. for $f \in L^2(G;\tau)$,

$$e^{-t\widetilde{\Delta}_\tau} f(g) = \int_G H_t^\tau(g^{-1}g') f(g') \mathrm{d}g'. \tag{5.7}$$

We denote by $(\mathcal{C}^q(G) \otimes \mathrm{End}(V_\tau))^{K\times K}$ the Harish-Chandra $L^q$-Schwartz space of functions on $G$ with values in $\mathrm{End}(V_\tau)$, defined as in [1, pp. 161–162], such that the covariance property (5.6) is satisfied.

**Theorem 5.1** *Let $t>0$. Then, for every $q>0$*

$$H_t^\tau \in (\mathcal{C}^q(G) \otimes \mathrm{End}(V_\tau))^{K\times K}.$$

*Proof* This is proved in [1, Proposition 2.4]. □

Let $R_\Gamma(H_t^\tau)$ be the trace class operator, induced by the right regular representation $R_\Gamma$ of $G$, acting on $C^\infty(G;\tau)$. In [1, p. 161], it is proved that

$$e^{-t\widetilde{\Delta}_\tau} = R_\Gamma(H_t^\tau).$$

We consider a unitary admissible representation $\pi$ of $G$ on a Hilbert space $\mathcal{H}_\pi$. We set

$$\widetilde{\pi}(H_t^\tau) = \int_G \pi(g) \otimes H_t^\tau(g) \mathrm{d}g.$$

This defines a trace class operator on $\mathcal{H}_\pi \otimes V_\tau$. By [1, pp. 160–161], relative to the splitting

$$\mathcal{H}_\pi \otimes V_\tau = (\mathcal{H}_\pi \otimes V_\tau)^K \oplus [(\mathcal{H}_\pi \otimes V_\tau)^K]^\perp,$$

$\widetilde{\pi}(H_t^\tau)$ has the form

$$\widetilde{\pi}(H_t^\tau) = \begin{pmatrix} \pi(H_t^\tau) & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.8}$$

with $\pi(H_t^\tau)$ acting on $(\mathcal{H}_\pi \otimes V_\tau)^K$. Then, it follows that

$$e^{-t(-\pi(\Omega)+\lambda_\tau)}\,\mathrm{Id} = \pi\left(H_t^\tau\right), \tag{5.9}$$

where Id denotes the identity on the space $(\mathcal{H}_\pi \otimes V_\tau)^K$ [1, Corollary 2.2]. We let

$$h_t^\tau(g) := \mathrm{tr}\; H_t^\tau(g).$$

We consider orthonormal bases $(\xi_n)$, $n \in \mathbb{N}$, $(e_j)$, $j = 1, \ldots, k$ of the vector spaces $\mathcal{H}_\pi$, $V_\tau$, respectively, where $k := \dim(V_\tau)$. By (5.8) we have

$$\mathrm{Tr}\left(\pi(H_t^\tau)\right) = \mathrm{Tr}\left(\widetilde{\pi}\left(H_t^\tau\right)\right), \tag{5.10}$$

and

$$\begin{aligned}
\mathrm{Tr}\left(\widetilde{\pi}\left(H_t^\tau\right)\right) &= \sum_n \sum_j \langle \widetilde{\pi}\left(H_t^\tau\right)\left(\xi_n \otimes e_j\right), \left(\xi_n \otimes e_j\right)\rangle \\
&= \sum_n \sum_j \int_G \langle \pi(g)\xi_n, \xi_n\rangle \langle H_t^\tau(g)e_j, e_j\rangle \mathrm{d}g \\
&= \sum_n \int_G \langle \pi(g)\xi_n, \xi_n\rangle h_t^\tau(g)\mathrm{d}g \\
&= \sum_n \langle \pi(h_t^\tau)\xi_n, \xi_n\rangle \\
&= \mathrm{Tr}\,\pi(h_t^\tau).
\end{aligned} \tag{5.11}$$

Hence, if we combine Eqs. (5.9), (5.10) and (5.11), we get

$$\mathrm{Tr}\,\pi(h_t^\tau) = e^{-t(-\pi(\Omega)+\lambda_\tau)} \dim(\mathcal{H}_\pi \otimes V_\tau)^K. \tag{5.12}$$

We specify now the unitary representation $\pi$ of $G$. We consider the unitary principal series representation $\pi_{\sigma,\lambda}$, defined in Sect. 2. Our goal is to compute the *Fourier transform* $\Theta_{\sigma,\lambda}(h_t^\tau)$ of $h_t^\tau$,

$$\Theta_{\sigma,\lambda}\left(h_t^\tau\right) := \mathrm{Tr}\,\pi_{\sigma,\lambda}\left(h_t^\tau\right).$$

**Proposition 5.2** *For $\sigma \in \widehat{M}$ and $\lambda \in \mathbb{R}$, let $\Theta_{\sigma,\lambda}$ be the character of $\pi_{\sigma,\lambda}$. Let $\tau \in \widehat{K}$. Then,*

$$\Theta_{\sigma,\lambda}\left(h_t^\tau\right) = e^{-t\left(-\pi_{\sigma,\lambda}(\Omega)+\lambda_\tau\right)}\left[\tau\,|_M\!: \sigma\right]. \tag{5.13}$$

*Proof* We have

$$\Theta_{\sigma,\lambda}(h_t^\tau) = e^{-t(-\pi_{\sigma,\lambda}(\Omega)+\lambda_\tau)} \dim(\mathcal{H}_{\pi_{\sigma,\lambda}} \otimes V_\tau)^K = e^{-t(-\pi_{\sigma,\lambda}(\Omega)+\lambda_\tau)}[\pi_{\sigma,\lambda} : \check{\tau}],$$

where $\check{\tau}$ denotes the contragredient representation of $\tau$. By Frobenius reciprocity (see [12, p. 208]), one has

$$[\pi_{\sigma,\lambda}|_K : \tau] = \sum_{\omega \in (M\cap K)\widehat{}} n_\omega[\tau\,|_{M\cap K}\!: \omega],$$

where $n_\omega$ are positive integers such that $\sigma|_{M\cap K} = \sum_{\omega\in(M\cap K)\widehat{}} n_\omega \omega$. In our case $M \subset K$ and therefore $M \cap K = M$. Hence,

$$[\pi_{\sigma,\lambda}|_K : \tau] = [\tau\,|_M\!: \sigma].$$

We have

$$\begin{aligned}\Theta_{\sigma,\lambda}(h_t^\tau) &= e^{-t(-\pi_{\sigma,\lambda}(\Omega)+\lambda_\tau)}[\pi_{\sigma,\lambda} : \check{\tau}]\\ &= e^{-t(-\pi_{\sigma,\lambda}(\Omega)+\lambda_\tau)}[\pi_{\sigma,\lambda} : \tau] = e^{-t(-\pi_{\sigma,\lambda}(\Omega)+\lambda_\tau)}[\tau\ |_M : \sigma].\end{aligned}$$

The assertion follows. □

We pass now to $X = \Gamma\backslash\widetilde{X}$. We consider the locally homogeneous vector bundle

$$E_\tau := \Gamma\backslash\widetilde{E}_\tau \to X.$$

Let $E_\chi$ be the flat vector bundle over $X$. We want to derive a trace formula for the heat operator $e^{-t\Delta^\sharp_{\tau,\chi}}$. By Lemma 2.4 in [23], $e^{-t\Delta^\sharp_{\tau,\chi}}$ is an integral operator with smooth kernel and of trace class.

Indeed, since $\Delta^\sharp_{\tau,\chi}$ is a second-order elliptic differential operator, by Lemma 8.6 (see "Appendix"), its spectrum is discrete and for every $\varepsilon \in (0, \frac{\pi}{2})$ there exists $R > 0$ such that its spectrum is contained in the set $B(0, R) \cup L_{[-\varepsilon,\varepsilon]} \subset \mathbb{C}$. Let $\theta$ be an Agmon angle for $\Delta^\sharp_{\tau,\chi}$. Let $V_0$ be the eigenspace of the eigenvalue 0. Let $V_0'$ be the complementary subspace of $V_0$. We define the operator $\Delta^\sharp_{\tau,\chi,1}$ by $\Delta^\sharp_{\tau,\chi,1} = \Delta^\sharp_{\tau,\chi}|_{V_0'}$. We define the operator ${\Delta^\sharp_{\tau,\chi,1}}^{1/2}$ as in [23, p. 8]. Let $\phi(\lambda) = e^{-t\lambda^2}$, $\lambda \in \mathbb{C}$. We define the contour $\Gamma'$ as in [23, p. 10]. The operator $\phi({\Delta^\sharp_{\tau,\chi,1}}^{1/2})$ is defined by

$$\phi\left({\Delta^\sharp_{\tau,\chi,1}}^{1/2}\right) = \frac{i}{2\pi}\int_{\Gamma'} \phi(\lambda)\big({\Delta^\sharp_{\tau,\chi,1}}^{1/2} - \lambda\,\mathrm{Id}\big)^{-1} d\lambda.$$

Let $\Pi_0$ be the projection onto the 0-eigenspace $V_0$ of the eigenvalue 0. Let $N = \Delta^\sharp_{\tau,\chi}\Pi_0$. Put $U(t; N) := \sum_{k=0}^{m} \frac{(-1)^k t^k}{k!} N^k$, where $m \in \mathbb{N}$ such that $N^m = 0$ (see [23, p. 11]). Let $\widehat{\phi}$ be the Fourier transform of $\phi|_{\mathbb{R}}$. Put

$$\phi\left(N^{1/2}\right) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{\phi}(t) U(t; N).$$

Then, by the definition of $U(t; N)$, we get $\phi(N^{1/2}) = \sum_{k=0}^{m} \frac{i^k \phi^k(0)}{k!} N^k$. As in [23, Eq. (2.24)] we define

$$e^{-t\Delta^\sharp_{\tau,\chi}} = \phi\left({\Delta^\sharp_{\tau,\chi}}^{1/2}\right) = \phi\left({\Delta^\sharp_{\tau,\chi,1}}^{1/2}\right)(\mathrm{Id} - \Pi_0) + \phi\left(N^{1/2}\right)\Pi_0.$$

One can proceed as in the proof of Lemma 2.4 in [23] to prove that $\phi(N^{1/2})\Pi_0$ is an integral operator with smooth kernel. To prove that $\phi({\Delta^\sharp_{\tau,\chi,1}}^{1/2})(\mathrm{Id} - \Pi_0)$ is an integral operator with smooth kernel, one can proceed as in the proof of Lemma 2.4 in [23], using the fact that the function $\lambda^{2(k+l)}\phi(\lambda) = \lambda^{2(k+l)}e^{-t\lambda^2}$ is a rapidly decreasing function of $\lambda$ on $\Gamma'$. We apply now Lidskii's theorem, which gives an expression for the trace of a trace class operator in terms of its eigenvalues. By [27, Theorem 3.7], we have

$$\operatorname{Tr} e^{-t\Delta^\sharp_{\tau,\chi}} = \sum_{\lambda_j \in \operatorname{spec}\left(\Delta^\sharp_{\tau,\chi}\right)} m(\lambda_j) e^{-t\lambda_j}, \tag{5.14}$$

where $m(\lambda_j)$ is as in Definition 8.7 in "Appendix."

We use the initial value problem for the wave equation to study the integral operator $\phi({\Delta^{\sharp}_{\tau,\chi}}^{1/2})$. Let $t \in \mathbb{R}$ and $f \in C^{\infty}(X, E_\tau \otimes E_\chi)$. Let $u(t; f)$ be the solution of the following initial value problem for the wave equation

$$\frac{\partial^2}{\partial t^2}u(t; f) + {\Delta_{\tau,\chi}}^{\sharp}u(t; f) = 0, \quad u(0; f) = f, \quad u_t(0; f) = 0.$$

We put $u_1(t; f) := u(-t; f)$. Then, we observe that $u_1(0; f) = u(0; f) = f$ and ${u_1}_t(0; f) = -u_t(0; f) = 0$. In addition,

$$\frac{\partial^2}{\partial t^2}u_1(t; f) + {\Delta_{\tau,\chi}}^{\sharp}u_1(t; f) = \frac{\partial^2}{\partial t^2}u(-t; f) + {\Delta_{\tau,\chi}}^{\sharp}u(-t; f) = 0$$

Hence, $u$ and $u_1$ solve the initial value problem for the wave equation. By uniqueness of the solution it follows that $u(t; f) = u(-t; f)$. As in the proof of Proposition 3.1 in [23], the wave equation can be replaced by a symmetric first-order hyperbolic system $\frac{\partial}{\partial t}(u_1, u_1) = L(u_1, u_2)$ with initial condition (see [23, (3.3)] for further details). Here, $L$ is a pseudodifferential operator of order 1. For $t \geq 0$, we have

$$\begin{aligned}\frac{\partial}{\partial t}\|u(t; f)\| &= (u'(t; f), u(t; f)) + (u(t; f), u'(t; f)) \\ &= ((L + L^*)u(t; f), u(t; f)).\end{aligned}$$

Since $(L + L^*)$ is a pseudodifferential operator of order 0, there exists a positive constant $c$ such that $((L + L^*)u(t; f), u(t; f)) \leq c\|u(t; f)\|$. Hence, $\frac{\partial}{\partial t}\|u(t; f)\| \leq c\|u(t; f)\|$. Applying Gronwall's inequality [29, Lemma 2.2], we get $\|u(t; f)\| \leq e^{ct}\|u(0; f)\|$, for $t \geq 0$. Since $u(t; f) = u(-t; f)$, this holds for all $t \in \mathbb{R}$. Let $s \in \mathbb{R}$. Let $\|\cdot\|_s$ be the Sobolev norm of the Sobolev space $H^s$. Then, since $\|u(t; f)\|_s^2 = \|(\Delta^{\sharp}_{\tau,\chi} + \mathrm{Id})^{s/2}u(t; f)\|^2$, we can argue as above and use the Gronwall's inequality (see [29, p. 74]). We get

$$\|u(t; f)\|_s \leq e^{ct}\|u(0; f)\|_s, \quad \forall t \in \mathbb{R}.$$

Let now $\hat{\phi}$ be the Fourier transform of $\phi|_{\mathbb{R}}$. We want to prove that for $f \in C^{\infty}(X, E_\tau \otimes E_\chi)$,

$$\phi\left({\Delta^{\sharp}_{\tau,\chi}}^{1/2}\right) f = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{\phi}(w)u(w; f)\mathrm{d}w.$$

The integral above is absolutely convergent, since for $t \geq 0, \hat{\phi}(w) = \frac{e^{-w^2/4t}}{\sqrt{2t}}$ and $\|u(w; f)\| \leq e^{cw}\|u(0; f)\|$. As in the proof of Proposition 3.2 in [23], we consider $c \geq 0$, such that the spectrum of ${\Delta^{\sharp}_{\tau,\chi}}^{1/2}$ is contained in $\mathrm{Re}(z) > 0$. For $\sigma > 0$, we define the operator $\cos(t{\Delta^{\sharp}_{\tau,\chi}}^{1/2})e^{-\sigma(\Delta^{\sharp}_{\tau,\chi}+c)}$ by the well-defined integral

$$\cos\left(w{\Delta^{\sharp}_{\tau,\chi}}^{1/2}\right)e^{-\sigma\left(\Delta^{\sharp}_{\tau,\chi}+c\right)} = \frac{i}{2\pi}\int_{\Gamma'}\cos(w\lambda)e^{-\sigma(\lambda^2+c)}\left({\Delta^{\sharp}_{\tau,\chi}}^{1/2} - \lambda\right)^{-1}\mathrm{d}\lambda$$

(see [23, p. 15]). For $f \in C^{\infty}(X, E_\tau \otimes E_\chi)$, we set

$$u(w; \sigma, f) := \cos\left(w{\Delta^{\sharp}_{\tau,\chi}}^{1/2}\right)e^{-\sigma\left(\Delta^{\sharp}_{\tau,\chi}+c\right)}f.$$

Then, $u(w; \sigma, f)$ is the solution of the initial value problem for the wave equation with initial condition $u(0; \sigma, f) = e^{-\sigma(\Delta^{\sharp}_{\tau,\chi}+c)}f$. Moreover, $u(w; f) - u(w; \sigma, f)$ is the solution of the initial value problem for the wave equation with initial condition $f - e^{-\sigma(\Delta^{\sharp}_{\tau,\chi}+c)} f$. Hence, by the discussion above

$$\|u(w; f) - u(w; \sigma, f)\|_s \leq e^{cw} \|f - e^{-\sigma(\Delta^{\sharp}_{\tau,\chi}+c)} f\|_s, \quad \forall w \in \mathbb{R}.$$

We use this inequality as in [23, p. 16] to prove that

$$\lim_{\sigma\to 0} \|u(w; f) - u(w; \sigma, f)\|_s = 0,$$

and

$$\phi\left({\Delta^{\sharp}_{\tau,\chi}}^{1/2}\right) f = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{\phi}(w) u(w; f) \mathrm{d}w.$$

In addition, if $0 \in \operatorname{spec}(\Delta^{\sharp}_{\tau,\chi})$, one can proceed as in [23, pp. 16–17], to obtain the same equality.

We want now to consider the pullback of the solution of the wave equation to the universal covering $\widetilde{X}$. Let $\widetilde{u}(w; f) = \widetilde{u}(w, \widetilde{x}, f)$ and $\widetilde{f}$ be the pullback to $\widetilde{X}$ of $u(w; f) = u(w, x, f)$ and $f$, respectively. Then, $\widetilde{u}(w, \widetilde{x}, f) = u(w, \widetilde{x}, \widetilde{f})$ [23, Eq. (3.15)]. Let $x, y$ be in $X$ and $\widetilde{x}, \widetilde{y}$ be the lifts of $x, y$ to $\widetilde{X}$, respectively.

Let $H^{\tau,\chi}_t$ be the kernel function of the integral operator $e^{-t\Delta^{\sharp}_{\tau,\chi}}$. It is a smooth section of $(E_\tau \otimes E_\chi) \boxtimes (E_\tau \otimes E_\chi)^*$. Let $H^\tau_t$ be the kernel of $e^{-t\widetilde{\Delta}_\tau}$.

Let $\chi_T \in C^\infty_c(\mathbb{R})$ such that $\chi = 1$ on $[-T, T]$. Let $\widehat{\phi_T}$ be the Fourier transform of $\phi_T := \phi\chi_T$. Then, by [23, Proposition 3.3], we have

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{\phi_T}(w) u(w, \widetilde{x}, \widetilde{f}) \mathrm{d}w = \int_{\widetilde{X}} H_{\phi_T}(\widetilde{x}, \widetilde{y}) \widetilde{f}(\widetilde{y}) \mathrm{d}\widetilde{y},$$

where $H_{\phi_T} \in C^\infty(\widetilde{X} \times \widetilde{X}, \operatorname{Hom}(\widetilde{E}_\tau \otimes \widetilde{E}_\chi, \widetilde{E}_\tau \otimes \widetilde{E}_\chi))$. As $T \to \infty$, the equation above becomes

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{\phi}(w) u(w, \widetilde{x}, \widetilde{f}) \mathrm{d}w = \int_{\widetilde{X}} H_{\phi}(\widetilde{x}, \widetilde{y}) \widetilde{f}(\widetilde{y}) \mathrm{d}\widetilde{y},$$

where $H_\phi$ is the kernel of the operator $\phi({\Delta^{\sharp}_{\tau,\chi}}^{1/2})$ lifted to $\widetilde{X}$, i.e.,

$$H_\phi(\widetilde{x}, \widetilde{y}) = H^\tau_t(\widetilde{x}, \widetilde{y}) \otimes \operatorname{Id}_{V_\chi}.$$

Let $F$ be a fundamental domain of $\Gamma$. Let $L^2(\widetilde{X}, \widetilde{E}_\tau \otimes \widetilde{E}_\chi)^\Gamma$ be the space of sections $\widetilde{f}$ of $\widetilde{E}_\tau \otimes \widetilde{E}_\chi$ such that $\widetilde{f}(\gamma\widetilde{x}) = \chi(\gamma)\widetilde{f}(\widetilde{x})$, $\forall \gamma \in \Gamma, \widetilde{x} \in \widetilde{X}$. For $f \in L^2(X, E_\tau \otimes E_\chi) \cong L^2(\widetilde{X}, \widetilde{E}_\tau \otimes \widetilde{E}_\chi)^\Gamma$, we have

$$\begin{aligned} e^{-t\Delta^{\sharp}_{\tau,\chi}} f(x) &= \int_X H^{\tau,\chi}_t(x, y) f(y) \mathrm{d}y \\ &= \int_{\widetilde{X}} \left(H^\tau_t(\widetilde{x}, \widetilde{y}) \otimes \operatorname{Id}_{V_\chi}\right) \widetilde{f}(\widetilde{y}) \mathrm{d}\widetilde{y} \\ &= \sum_{\gamma \in \Gamma} \int_F \left(H^\tau_t(\widetilde{x}, \gamma\widetilde{y}) \otimes \chi(\gamma) \operatorname{Id}_{V_\chi}\right) \widetilde{f}(\widetilde{y}) \mathrm{d}\widetilde{y}. \end{aligned} \tag{5.15}$$

$H^\tau_t$ belongs to the Harish-Chandra $L^q$-Schwartz space. Hence, since $\widetilde{f}$ is bounded, the integral in the second line of (5.15) is well defined. Moreover, we can interchange summation and integration and get

$$H_t^{\tau,\chi}(x,x') = \sum_{\gamma\in\Gamma} H_t^\tau\left(g^{-1}\gamma g'\right)\otimes\chi(\gamma),$$

where $x = \Gamma g$, $x' = \Gamma g'$ and $g, g' \in G$. By [23, Proposition 4.1], we have the following proposition.

**Proposition 5.3** *Let $E_\chi$ be a flat vector bundle over $X = \Gamma\backslash\widetilde{X}$ associated with a finite dimensional representation $\chi : \Gamma \to \mathrm{GL}(V_\chi)$ of $\Gamma$. Let $\Delta^\sharp_{\tau,\chi}$ be the twisted Bochner–Laplace operator acting on $C^\infty(X, E_\tau \otimes E_\chi)$. Then,*

$$\begin{aligned}\mathrm{Tr}\left(e^{-t\Delta^\sharp_{\tau,\chi}}\right) &= \sum_{\lambda_j\in\mathrm{spec}\left(\Delta^\sharp_{\tau,\chi}\right)} m(\lambda_j)e^{-t\lambda_j}\\ &= \sum_{\gamma\in\Gamma}\mathrm{tr}\,\chi(\gamma)\int_{\Gamma\backslash G}\mathrm{tr}\ H_t^\tau\left(g^{-1}\gamma g\right)\mathrm{d}\dot{g}. \qquad (5.16)\end{aligned}$$

The action of the non-trivial element $w$ of $W_A$ on $R(M)$ determines an eigenspace decomposition of $R(M)$ into $R(M)^+$ and $R(M)^-$. The subspaces $R(M)^\pm$ are the $(\pm1)$-eigenspaces of w.

**Proposition 5.4** 1. *The map $i^*$ is a bijection between $R(K)$ and $R(M)^+$.*

2. *Let $\sigma \in \widehat{M}$ be of highest weight as in (2.5) and assume that $\nu_n > 0$. Then, there exists a unique element $\tau(\sigma) \in \widehat{K}$, such that*

$$\sigma - w\sigma = (s^+ - s^-)i^*(\tau(\sigma)), \qquad (5.17)$$

*where $s^+, s^-$ are the half-spin representations of $M$. In addition, $\tau(\sigma)\otimes s$ splits into two representations $\tau^+(\sigma)$ and $\tau^-(\sigma)$,*

$$\tau(\sigma)\otimes s = \tau^+(\sigma)\oplus\tau^-(\sigma), \qquad (5.18)$$

*such that*

$$\sigma + w\sigma = i^*(\tau^+(\sigma) - \tau^-(\sigma)). \qquad (5.19)$$

*Proof* See [2, Proposition 1.1]. □

Let $\tau_\sigma \in R(K)$ defined by

$$\tau_\sigma := \tau^+(\sigma) - \tau^-(\sigma). \qquad (5.20)$$

As in the proof of Proposition 1.1 in [2, p. 22] (see also [24, Proposition 2.3]), $\tau^+(\sigma)$, $\tau^-(\sigma)$ decompose into irreducible representations of K. Then, there exist unique integers $m_\tau(\sigma) \in \{-1, 0, 1\}$, which are equal to zero except for finitely many $\tau \in \widehat{K}$, such that ,

– if $\sigma$ is Weyl invariant,

$$\sigma = \sum_{\tau\in\widehat{K}} m_\tau(\sigma)i^*(\tau); \qquad (5.21)$$

– if $\sigma$ is non-Weyl invariant,

$$\sigma + w\sigma = \sum_{\tau\in\widehat{K}} m_\tau(\sigma)i^*(\tau). \qquad (5.22)$$

We define a locally homogeneous vector bundle $E(\sigma)$ associated with $\sigma$ in the following way.

$$E(\sigma) = \bigoplus_{\substack{\tau\in\widehat{K}\\ m_\tau(\sigma)\neq 0}} E_\tau,$$

where $E_\tau$ is the locally homogeneous vector bundle associated with $\tau \in \widehat{K}$. The vector bundle $E(\sigma)$ has a grading $E(\sigma) = E(\sigma)^+ \oplus E(\sigma)^-$. This grading is defined exactly by the positive or negative sign of $m_\tau(\sigma)$. Let $\widetilde{E}(\sigma)$ be the pullback of $E(\sigma)$ to $\widetilde{X}$. Then,

$$\widetilde{E}(\sigma) = \bigoplus_{\substack{\tau\in\widehat{K}\\ m_\tau(\sigma)\neq 0}} \widetilde{E}_\tau.$$

We consider now the lift $\widetilde{\Delta}_\tau$ of the Bochner–Laplace operator $\Delta_\tau$, associated with $\tau \in \widehat{K}$, to $\widetilde{X}$, acting on smooth sections of $\widetilde{E}_\tau$. Recall $\widetilde{\Delta}_\tau = -R(\Omega) + \lambda_\tau \operatorname{Id}$. We put

$$\widetilde{A}_\tau := \widetilde{\Delta}_\tau - \lambda_\tau \operatorname{Id}. \tag{5.23}$$

Hence, the operator $\widetilde{A}_\tau$ acts like $-R(\Omega)$ on the space of smooth sections of $\widetilde{E}_\tau$. It is an elliptic formally self-adjoint operator of second order. By [4], it is an essentially self-adjoint operator. Its self-adjoint extension will be also denoted by $\widetilde{A}_\tau$. If $\widetilde{\Delta}^{\sharp}_{\tau,\chi}$ is the lift of the twisted Bochner–Laplace operator to the universal covering $\widetilde{X}$, we get the operator $\widetilde{A}^{\sharp}_{\tau,\chi}$ acting on the space $C^\infty(\widetilde{X}, \widetilde{E}_\tau \otimes \widetilde{E}_\chi)$, defined by

$$\widetilde{A}^{\sharp}_{\tau,\chi} = \widetilde{A}_\tau \otimes \operatorname{Id}_{V_\chi}. \tag{5.24}$$

We pass to $X = \Gamma\backslash\widetilde{X}$. We put

$$c(\sigma) := -|\rho|^2 - |\rho_m|^2 + |\nu_\sigma + \rho_m|^2, \tag{5.25}$$

where $\nu_\sigma$ is as in (2.5) and $\rho$, $\rho_m$ are defined by (2.3) and (2.4), respectively. We define the operator $A^{\sharp}_\chi(\sigma)$ acting on $C^\infty(X, E(\sigma) \otimes E_\chi)$ by

$$A^{\sharp}_\chi(\sigma) := \bigoplus_{m_\tau(\sigma)\neq 0} A^{\sharp}_{\tau,\chi} + c(\sigma). \tag{5.26}$$

Obviously, $A^{\sharp}_\chi(\sigma)$ preserves the grading. It is an elliptic operator of order two. However, the situation is now different, because it is not a self-adjoint operator anymore.

We deal first with the heat operator $e^{-tA^{\sharp}_{\tau,\chi}}$ generated by the operator $A^{\sharp}_{\tau,\chi}$. Since $A^{\sharp}_{\tau,\chi}$ is induced by $\Delta^{\sharp}_{\tau,\chi}$, it is an integral operator with smooth kernel. By Proposition 5.3, its trace is given by

$$\operatorname{Tr}(e^{-tA^{\sharp}_{\tau,\chi}}) = \sum_{\gamma\in\Gamma} \operatorname{tr}\chi(\gamma) \int_{\Gamma\backslash G} \operatorname{tr} Q^\tau_t(g^{-1}\gamma g)\, d\dot{g}, \tag{5.27}$$

where $Q_t^\tau \in (\mathcal{C}^q(G) \otimes \mathrm{End}(V_\tau))^{K\times K}$ is the kernel associated with the operator $e^{-t\widetilde{A}_\tau}$. We put

$$\begin{aligned} q_t^\tau &= \mathrm{tr}\ Q_t^\tau(g); \\ q_t^\sigma &= \sum_{\tau\in\widehat{K}} m_\tau(\sigma) q_t^\tau; \qquad (5.28) \\ K(t;\sigma) &= \sum_{\tau\in\widehat{K}} m_\tau(\sigma)\,\mathrm{Tr}(e^{-tA^\sharp_{\tau,\chi}}). \qquad (5.29) \end{aligned}$$

We proceed as in [31, pp. 177–178] to express the integral on the right-hand side of (5.27) in terms of global characters. We have

$$\begin{aligned} K(t;\sigma) =& \dim(V_\chi)\,\mathrm{Vol}(X) q_t^\sigma(e) \\ &+ \frac{1}{2\pi}\sum_{[\gamma]\neq e}\frac{l(\gamma)\,\mathrm{tr}(\chi(\gamma))}{n_\Gamma(\gamma)D(\gamma)}\sum_{\sigma\in\widehat{M}}\overline{\mathrm{tr}\,\sigma(m_\gamma)}\int_{\mathbb{R}}\Theta_{\sigma,\lambda}(q_t^\sigma)e^{-il(\gamma)\lambda}\mathrm{d}\lambda. \end{aligned} \qquad (5.30)$$

We continue analyzing the trace formula above in terms of characters. For the identity contribution we have

$$(q_t^\sigma)(e) = \sum_{\sigma\in\widehat{M}}\int_{\mathbb{R}}\Theta_{\sigma,\lambda}\left(q_t^\sigma\right)P_\sigma(i\lambda)\mathrm{d}\lambda, \qquad (5.31)$$

where $P_\sigma(i\lambda)$ denotes the Plancherel polynomial, defined in Sect. 2. By (5.28), we get

$$\Theta_{\sigma,\lambda}(q_t^\sigma) = \sum_{\tau\in\widehat{K}} m_\tau(\sigma)\Theta_{\sigma,\lambda}(q_t^\tau). \qquad (5.32)$$

By Proposition 5.2,

$$\Theta_{\sigma,\lambda}(q_t^\tau) = e^{-t(-\pi_{\sigma,\lambda}(\Omega))}[\tau|_M : \sigma]. \qquad (5.33)$$

The term $\lambda_\tau$ does not occur here, since our operator $A^\sharp_{\tau,\chi}$ is induced by the operator $A_\tau = \Delta_\tau - \lambda_\tau\,\mathrm{Id}$. We recall also

$$\pi_{\sigma,\lambda}(\Omega) = -\lambda^2 + c(\sigma) \qquad (5.34)$$

(see [12, Proposition 8.22]). Combining Eqs. (5.32), (5.33) and (5.34) we get

$$\Theta_{\sigma,\lambda}(q_t^\sigma) = \sum_{\tau\in\widehat{K}} m_\tau(\sigma)e^{-t(\lambda^2-c(\sigma))}[\tau|_M : \sigma]. \qquad (5.35)$$

Equivalently, for $\sigma, \sigma' \in \widehat{M}$

$$\Theta_{\sigma',\lambda}(q_t^\sigma) = e^{tc(\sigma)}e^{-t\lambda^2}\left[\sum_{\tau\in\widehat{K}} m_\tau(\sigma) i^*(\tau) : \sigma'\right].$$

Hence, by (5.21) and (5.22), we have

$$\begin{aligned} \Theta_{\sigma',\lambda}(q_t^\sigma) &= e^{tc(\sigma)}e^{-t\lambda^2}, \quad \text{if } \sigma' \in \{\sigma, w\sigma\}, \\ \Theta_{\sigma',\lambda}(q_t^\sigma) &= 0, \quad \text{if } \sigma' \notin \{\sigma, w\sigma\}. \end{aligned} \qquad (5.36)$$

If we insert (5.31) and (5.36) in (5.30), we obtain

– If $\sigma$ is Weyl invariant,

$$K(t;\sigma)=e^{tc(\sigma_k)}\Bigg(\dim(V_\chi)\operatorname{Vol}(X)\int_{\mathbb{R}}e^{-t\lambda^2}P_\sigma(i\lambda)d\lambda + \sum_{[\gamma]\neq[e]}\frac{l(\gamma)}{n_\Gamma(\gamma)}L(\gamma;\sigma,\chi)\frac{e^{-l(\gamma)^2/4t}}{(4\pi t)^{1/2}}\Bigg);$$

– If $\sigma$ is non-Weyl invariant,

$$K(t;\sigma)=e^{tc(\sigma_k)}\Bigg(2\dim(V_\chi)\operatorname{Vol}(X)\int_{\mathbb{R}}e^{-t\lambda^2}P_\sigma(i\lambda)d\lambda + \sum_{[\gamma]\neq e}\frac{l(\gamma)}{n_\Gamma(\gamma)}L(\gamma;\sigma+w\sigma,\chi)\frac{e^{-l(\gamma)^2/4t}}{(4\pi t)^{1/2}}\Bigg),$$

where

$$L(\gamma;\sigma,\chi)=\frac{\operatorname{tr}(\chi(\gamma)\otimes\sigma(m_\gamma))e^{-|\rho|l(\gamma)}}{\det(\operatorname{Id}-\operatorname{Ad}(m_\gamma a_\gamma)_{\overline{n}})}. \tag{5.37}$$

By the definition of the operator $A^\sharp_\chi(\sigma)$, we get the following theorem.

**Theorem 5.5** *Let $\sigma\in\widehat{M}$.*

– *If $\sigma$ is Weyl invariant,*

$$\operatorname{Tr}(e^{-tA^\sharp_\chi(\sigma)})=\dim(V_\chi)\operatorname{Vol}(X)\int_{\mathbb{R}}e^{-t\lambda^2}P_\sigma(i\lambda)d\lambda + \sum_{[\gamma]\neq e}\frac{l(\gamma)}{n_\Gamma(\gamma)}L(\gamma;\sigma,\chi)\frac{e^{-l(\gamma)^2/4t}}{(4\pi t)^{1/2}}; \tag{5.38}$$

– *If $\sigma$ is non-Weyl invariant,*

$$\operatorname{Tr}(e^{-tA^\sharp_\chi(\sigma)})=2\dim(V_\chi)\operatorname{Vol}(X)\int_{\mathbb{R}}e^{-t\lambda^2}P_\sigma(i\lambda)d\lambda + \sum_{[\gamma]\neq e}\frac{l(\gamma)}{n_\Gamma(\gamma)}L(\gamma;\sigma+w\sigma,\chi)\frac{e^{-l(\gamma)^2/4t}}{(4\pi t)^{1/2}}, \tag{5.39}$$

*where $L(\gamma;\sigma,\chi)$ is as in (5.37).*

## 6 The twisted Dirac operator

### 6.1 Definition

We consider $\sigma\in\widehat{M}$ with highest weight $\nu_\sigma$ as in (2.5). Recall that $\nu_n$ denotes the last coordinate of $\nu_\sigma$. Throughout this section we consider $\sigma\in\widehat{M}$ being non-Weyl invariant. Recall also from Sect. 2 that $K=\operatorname{Spin}(d)$ and $M=\operatorname{Spin}(d-1)$ and $(s,S)$ is the spin representation of $K$. Since $d-1$ is an even integer, $s$ splits into two irreducible half-spin representations $(s^+,S^+)$, $(s^-,S^-)$ of $M$. Let $\operatorname{Cl}(\mathfrak{p})$ be the Clifford algebra of $\mathfrak{p}$ with respect to $\langle\cdot,\cdot\rangle$, defined by (2.1), restricted to $\mathfrak{p}$. Let $\cdot\colon \mathfrak{p}\otimes S\to S$ be the Clifford multiplication on $\mathfrak{p}\otimes S$.

Let $L$ be a spinor bundle, equipped with a connection $\nabla$. The Dirac operator $D\colon C^\infty(X,L)\to C^\infty(X,L)$ is defined as

$$D : C^\infty(X,L)\xrightarrow{\nabla} C^\infty(X,T^*X\otimes L)\xrightarrow{g} C^\infty(X,TX\otimes L)\xrightarrow{\cdot} C^\infty(X,L),$$

where we identify $TX\cong T^*X$ using the Riemannian metric $g$, and $\cdot$ denotes the Clifford multiplication as above. Locally, it can be described as

$$Df\equiv\sum_{i=1}^{d} e_i\cdot\nabla_{e_i}f,$$

where $(e_1,\ldots,e_d)$ is a local orthonormal frame for $T_xX$, $x\in X$. The operator $D$ is an elliptic formally self-adjoint operator of first order [14, Lemma 5.1, Proposition 5.3]. We want to define the Dirac operator acting on smooth sections of vector bundles associated with the representations $\sigma$ of $M$ and arbitrary representations $\chi$ of $\Gamma$.

Let $\tau(\sigma)\in\widehat{K}$ with representation space $V_{\tau(\sigma)}$ be as in Proposition 5.4. Let $\widetilde{E}_{\tau(\sigma)}$ be the homogeneous vector bundle over $\widetilde{X}$ given by

$$\widetilde{E}_{\tau(\sigma)}=G\times_{\tau(\sigma)}V_{\tau(\sigma)}\to\widetilde{X}.$$

Let $\nabla^{\tau(\sigma)}$ be the canonical connection in $\widetilde{E}_{\tau(\sigma)}$. By Proposition 5.4, there exist $\tau^+(\sigma)$, $\tau^-(\sigma)$ $\in R(K)$ such that

$$s\otimes\tau(\sigma)=\tau^+(\sigma)\oplus\tau^-(\sigma). \tag{6.1}$$

We define the representation $\tau_s(\sigma)$ of $K$ by

$$\tau_s(\sigma):=s\otimes\tau(\sigma), \tag{6.2}$$

with representation space $V_{\tau_s(\sigma)}=S\otimes V_{\tau(\sigma)}$. Let $E_s$ be the spinor bundle over $\widetilde{X}$ associated with $s$. The vector bundle $\widetilde{E}_{\tau_s(\sigma)}:=\widetilde{E}_{\tau(\sigma)}\otimes E_s$ over $\widetilde{X}$ carries a connection $\nabla^{\tau_s(\sigma)}$, defined by

$$\nabla^{\tau_s(\sigma)}:=\nabla^{\tau(\sigma)}\otimes 1+1\otimes\nabla.$$

We extend the Clifford multiplication by requiring that it acts on $V_{\tau_s(\sigma)}=S\otimes V_{\tau(\sigma)}$ as follows.

$$e\cdot(\phi\otimes\psi)=(e\cdot\phi)\otimes\psi,\quad e\in\mathrm{Cl}(\mathfrak{p}),\phi\in S,\psi\in V_{\tau(\sigma)}.$$

We define the Dirac operator $\widetilde{D}(\sigma)$ acting on $C^\infty(\widetilde{X},\widetilde{E}_{\tau_s(\sigma)})$ by

$$\widetilde{D}(\sigma)f=\sum_{i=1}^{d} e_i\cdot\nabla_{e_i}^{\tau_s(\sigma)}f,$$

where $(e_1,\ldots,e_d)$ is local orthonormal frame for $T_x\widetilde{X}$ and $f\in C^\infty(\widetilde{X},\widetilde{E}_{\tau_s(\sigma)})$. The space of smooth sections $C^\infty(\widetilde{X},\widetilde{E}_{\tau_s(\sigma)})$ can be identified with $C^\infty(G;\tau_s(\sigma))$ as in (5.1).

Let now $\chi:\Gamma\to\mathrm{GL}(V_\chi)$ be an arbitrary finite dimensional representation of $\Gamma$. Let $E_\chi$ be the associated flat vector bundle over $X$. Let $E_{\tau_s(\sigma)}:=\Gamma\backslash\widetilde{E}_{\tau_s(\sigma)}$ be the locally homogeneous vector bundle over $X$. We consider the product vector bundle $E_{\tau_s(\sigma)}\otimes E_\chi$ over $X$ and equip this bundle with the product connection $\nabla^{E_{\tau_s(\sigma)}\otimes E_\chi}$ defined by

$$\nabla^{E_{\tau_s(\sigma)}\otimes E_\chi}:=\nabla^{E_{\tau_s(\sigma)}}\otimes 1+1\otimes\nabla^{E_\chi}.$$

We consider the Clifford multiplication on $(V_{\tau_s(\sigma)} \otimes V_\chi)$ by requiring that it acts only on $V_{\tau_s(\sigma)}$, i.e.,

$$e \cdot (w \otimes v) = (e \cdot w) \otimes v, \quad e \in \mathrm{Cl}(\mathfrak{p}), w \in V_{\tau_s(\sigma)}, v \in V_\chi.$$

We introduce the twisted Dirac operator $D^\sharp_\chi(\sigma)$ associated with $\nabla^{E_{\tau_s(\sigma)} \otimes E_\chi}$. We consider then an open subset of $X$ such that $E_\chi|_U$ is trivial. Let also $(v_j)$, $j = 1, \dots, m$, be any base of flat sections of $E_\chi|_U$, where $m = \mathrm{rank}(E_\chi)$, and $\phi_j \in C^\infty(U, E_{\tau_s(\sigma)}|_U)$. Then,

$$E_{\tau_s(\sigma)} \otimes E_\chi|_U \cong \bigoplus_{j=1}^{m} E_{\tau_s(\sigma)}|_U,$$

and for each $\phi \in C^\infty(U, E_{\tau_s(\sigma)} \otimes E_\chi|_U)$,

$$\phi = \sum_{j=1}^{m} \phi_j \otimes v_j.$$

Then,

$$\nabla^{E_{\tau_s(\sigma)} \otimes E_\chi} \phi = \sum_{j=1}^{m} \nabla^{E_{\tau_s(\sigma)}} \phi_j \otimes v_j.$$

The twisted Dirac operator is described as follows.

$$\begin{aligned} D^\sharp_\chi(\sigma)\phi &= \sum_{i=1}^{d} e_i \cdot \nabla^{E_{\tau_s(\sigma)} \otimes E_\chi}_{e_i} \phi \\ &= \sum_{i=1}^{d} e_i \cdot \left( \sum_{j=1}^{m} \nabla^{E_{\tau_s(\sigma)}}_{e_i} \phi_j \otimes v_j \right) \\ &= \sum_{j=1}^{m} \sum_{i=1}^{d} e_i \cdot \nabla^{E_{\tau_s(\sigma)}}_{e_i} \phi_j \otimes v_j. \end{aligned} \tag{6.3}$$

We mention here that the local definition of the twisted Dirac operator is independent of the choice of the base of flat sections of $E_\chi|_U$, since the transition maps comparing flat sections are constant. We consider the pullbacks $\widetilde{E}_{\tau_s(\sigma)}$, $\widetilde{E}_\chi$ to $\widetilde{X}$ of $E_{\tau_s(\sigma)}$, $E_\chi$, respectively, then, $\widetilde{E}_\chi \cong \widetilde{X} \times V_\chi$. We have

$$C^\infty\left(\widetilde{X}, \widetilde{E}_{\tau_s(\sigma)} \otimes \widetilde{E}_\chi\right) \cong C^\infty\left(\widetilde{X}, \widetilde{E}_{\tau_s(\sigma)}\right) \otimes V_\chi.$$

With respect to this isomorphism, it follows from (6.3) that the lift $\widetilde{D}^\sharp_\chi(\sigma)$ of the twisted Dirac operator $D^\sharp_\chi(\sigma)$ to $\widetilde{X}$ is of the form

$$\widetilde{D}^\sharp_\chi(\sigma) = \widetilde{D}(\sigma) \otimes \mathrm{Id}_{V_\chi}. \tag{6.4}$$

### 6.2 The trace formula

The square of the twisted Dirac operator $D^\sharp_\chi(\sigma)^2$ acting on smooth sections of $E_{\tau_s(\sigma)} \otimes E_\chi$ is not a self-adjoint operator in general. Nevertheless, its principal symbol is given by

$$\sigma_{D^\sharp_\chi(\sigma)^2}(x, \xi) = \|\xi\|^2 \,\mathrm{Id}_{(V_{\tau_s(\sigma)} \otimes V_\chi)_x},$$

where $x \in X, \xi \in T_x^*X$. Therefore, it has nice spectral properties, i.e., its spectrum is discrete and contained in a translate of a positive cone $C \subset \mathbb{C}$ (Lemma 8.6, "Appendix"). Let $\widetilde{D}^{\sharp}_{\chi}(\sigma)$ be the lift of $D^{\sharp}_{\chi}(\sigma)$ to the universal covering $\widetilde{X}$. Since $\widetilde{D}^{\sharp}_{\chi}(\sigma) = \widetilde{D}(\sigma) \otimes \mathrm{Id}_{V_\chi}$,

$$\widetilde{D}^{\sharp}_{\chi}(\sigma)^2 = \widetilde{D}(\sigma)^2 \otimes \mathrm{Id}_{V_\chi} \tag{6.5}$$

We recall the definition of the operator $A^{\sharp}_{\chi}(\sigma)$ acting on $C^\infty(X, E(\sigma) \otimes E_\chi)$ from Sect. 5. We have

$$A^{\sharp}_{\chi}(\sigma) := \bigoplus_{m_\tau(\sigma)\neq 0} A^{\sharp}_{\tau,\chi} + c(\sigma),$$

where $c(\sigma)$ is as in (5.25). We consider the lift $\widetilde{A}^{\sharp}_{\chi}(\sigma)$ of $A^{\sharp}_{\chi}(\sigma)$ to the universal covering $\widetilde{X}$. Then,

$$\begin{aligned}\widetilde{A}^{\sharp}_{\chi}(\sigma) &= \bigoplus_{m_\tau(\sigma)\neq 0} \widetilde{A}^{\sharp}_{\tau,\chi} + c(\sigma)\\ &= \bigoplus_{m_\tau(\sigma)\neq 0} \big(\widetilde{A}_\tau + c(\sigma)\big) \otimes \mathrm{Id}_{V_\chi}. \end{aligned}\tag{6.6}$$

The Parthasarathy formula from [2, Eq. (1.11)] states

$$\widetilde{D}(\sigma)^2 = \bigoplus_{m_\tau(\sigma)\neq 0} \big(\widetilde{A}_\tau + c(\sigma)\big) \tag{6.7}$$

If we combine (6.5), (6.6), and (6.7) the Parthasarathy formula generalizes as

$$D^{\sharp}_{\chi}(\sigma)^2 = A^{\sharp}_{\chi}(\sigma). \tag{6.8}$$

We define the operators $e^{-tD^{\sharp}_{\chi}(\sigma)^2}$ and $D^{\sharp}_{\chi}(\sigma)e^{-t(D^{\sharp}_{\chi}(\sigma))^2}$ by

$$e^{-tD^{\sharp}_{\chi}(\sigma)^2} = \frac{i}{2\pi}\int_{\Gamma_{\theta,r_0}} e^{-t\lambda}\big(D^{\sharp}_{\chi}(\sigma)^2 - \lambda\,\mathrm{Id}\big)^{-1} d\lambda$$

$$D^{\sharp}_{\chi}(\sigma)e^{-tD^{\sharp}{}_{\chi}(\sigma)^2} = D^{\sharp}_{\chi}(\sigma)\left(\frac{i}{2\pi}\int_{\Gamma_{\theta,r_0}} e^{-t\lambda}\big(D^{\sharp}_{\chi}(\sigma)^2 - \lambda\,\mathrm{Id}\big)^{-1} d\lambda\right).$$

(see Eq. (8.1) in "Appendix" for further details). $D^{\sharp}_{\chi}(\sigma)^2$ is a second-order elliptic differential operator. One can prove that $e^{-tD^{\sharp}_{\chi}(\sigma)^2}$ is an integral operator with smooth kernel, as in Sect. 5, pp. 23–24, extending Lemma 2.4 in [23] for $\phi(\lambda) = e^{-t\lambda^2}$. Hence, $D^{\sharp}_{\chi}(\sigma)e^{-tD^{\sharp}_{\chi}(\sigma)^2}$ is an integral operator with smooth kernel $K_t^{\tau_s(\sigma),\chi}$. Let $K_t^{\tau_s(\sigma)}$ be the kernel of the operator $\widetilde{D}(\sigma)e^{-t\widetilde{D}(\sigma)^2}$. It belongs to the Harish-Chandra $L^q$-Schwartz space $(\mathcal{C}^q(G) \otimes \mathrm{End}(V_{\tau_s(\sigma)}))^{K\times K}$. Let $x$, $y$ be in $X$ and $\widetilde{x}$, $\widetilde{y}$ be the lifts of $x$, $y$ to $\widetilde{X}$, respectively. By (6.4)–(6.8), we get

$$K_t^{\tau_s(\sigma),\chi}(\widetilde{x},\widetilde{y}) = K_t^{\tau_s(\sigma)}(\widetilde{x},\widetilde{y}) \otimes \mathrm{Id}_{V_\chi}.$$

Let $F$ be a fundamental domain of $\Gamma$. We consider the space $L^2(\widetilde{X}, \widetilde{E}_{\tau_s(\sigma)} \otimes \widetilde{E}_\chi)^\Gamma$ of sections $\widetilde{f}$ of $\widetilde{E}_{\tau_s(\sigma)} \otimes \widetilde{E}_\chi$ such that $\widetilde{f}(\gamma\widetilde{x}) = \chi(\gamma)\widetilde{f}(\widetilde{x}), \forall \gamma \in \Gamma, \widetilde{x} \in \widetilde{X}$.

Arguing as in Sect. 5, pp. 24–26, we have for $f \in L^2(X, E_{\tau_s(\sigma)} \otimes E_\chi) \cong L^2(\widetilde{X}, \widetilde{E}_{\tau_s(\sigma)} \otimes \widetilde{E}_\chi)^\Gamma$

$$\begin{aligned} D^\sharp_\chi(\sigma) e^{-tD^\sharp_\chi(\sigma)^2} f(x) &= \int_X K_t^{\tau_s(\sigma),\chi}(x,y) f(y) \mathrm{d}y \\ &= \int_{\widetilde{X}} (K_t^{\tau_s(\sigma)}(\widetilde{x},\widetilde{y}) \otimes \mathrm{Id}_{V_\chi}) \widetilde{f}(\widetilde{y}) \mathrm{d}\widetilde{y} \\ &= \sum_{\gamma\in\Gamma} \int_F (K_t^{\tau_s(\sigma)}(\widetilde{x},\gamma\widetilde{y}) \otimes \chi(\gamma)\, \mathrm{Id}_{V_\chi}) \widetilde{f}(\widetilde{y}) \mathrm{d}\widetilde{y}. \end{aligned} \tag{6.9}$$

The integral in the second line of (6.9) is well defined, since $K_t^{\tau_s(\sigma)}$ belongs to the Harish-Chandra $L^q$-Schwartz space and $\widetilde{f}$ is bounded. In addition, since $K_t^{\tau_s(\sigma)}$ belongs to the Harish-Chandra $L^q$-Schwartz space, we can interchange summation and integration in the third line of (6.9) and get

$$K_t^{\tau_s(\sigma),\chi}(x,x') = \sum_{\gamma\in\Gamma} K_t^{\tau_s(\sigma)}(g^{-1}\gamma g') \otimes \chi(\gamma),$$

where $x = \Gamma g$, $x' = \Gamma g'$ and $g, g' \in G$. By [23, Proposition 4.1], we have

$$\mathrm{Tr}\left(D^\sharp_\chi(\sigma) e^{-tD^\sharp_\chi(\sigma)^2}\right) = \sum_{\gamma\in\Gamma} \mathrm{tr}\,\chi(\gamma) \int_{\Gamma\backslash G} \mathrm{tr}\; K_t^{\tau_s(\sigma)}(g^{-1}\gamma g) \mathrm{d}g.$$

We put

$$k_t^{\tau_s(\sigma)}(g) = \mathrm{tr}\; K_t^{\tau_s(\sigma)}(g). \tag{6.10}$$

By [31, pp. 177–178], we have

$$\begin{aligned} \mathrm{Tr}\left(D^\sharp_\chi(\sigma) e^{-tD^\sharp_\chi(\sigma)^2}\right) &= \dim(V_\chi)\, \mathrm{Vol}(X) (k_t^{\tau_s(\sigma)})(e) \\ &\quad + \frac{1}{2\pi} \sum_{[\gamma]\neq e} \frac{l(\gamma)\, \mathrm{tr}(\chi(\gamma))}{n_\Gamma(\gamma) D(\gamma)} \sum_{\sigma\in\widehat{M}} \overline{\mathrm{tr}\,\sigma(m_\gamma)} \int_{\mathbb{R}} \Theta_{\sigma,\lambda}(k_t^{\tau_s(\sigma)}) e^{-il(\gamma)\lambda} \mathrm{d}\lambda. \end{aligned}$$

We continue analyzing the trace formula above in terms of characters. We want to compute the *Fourier transform* $\Theta_{\sigma,\lambda}(k_t^{\tau_s(\sigma)})$ of $k_t^{\tau_s(\sigma)}$. Following [22] we let $(\pi, \mathcal{H}_\pi)$ be an unitary admissible representation of $G$ in a Hilbert space $\mathcal{H}_\pi$. We let $\mathcal{H}_\pi^\infty$ be the subspace of smooth vectors of $\mathcal{H}_\pi$. We set

$$\widetilde{\pi}\left(K_t^{\tau_s(\sigma)}\right) := \int_G \pi(g) \otimes K_t^{\tau_s(\sigma)}(g) \mathrm{d}g. \tag{6.11}$$

This defines a bounded trace class operator on $\mathcal{H}_\pi \otimes V_{\tau_s(\sigma)}$. Then, as in Sect. 5 (Eq. 5.11),

$$\mathrm{Tr}\left(\pi(k_t^{\tau_s(\sigma)})\right) = \mathrm{Tr}\left(\widetilde{\pi}\left(K_t^{\tau_s(\sigma)}\right)\right). \tag{6.12}$$

Let $(X_i)_{i=1}^d$ be an orthonormal basis of $\mathfrak{p}$. We consider the operator acting on $(\mathcal{H}_\pi^\infty \otimes V_{\tau_s(\sigma)})^K$, defined by

$$\widetilde{D}_{\tau_s(\sigma)}(\pi) := \sum_{i=1}^d X_i \cdot (\pi(X_i) \otimes \mathrm{Id}). \tag{6.13}$$

Arguing as in Sect. 5, we set

$$\widetilde{\pi}\left(H_t^{\tau_s(\sigma)}\right) = \int_G \pi(g) \otimes H_t^{\tau_s(\sigma)} \mathrm{d}g.$$

$H_t^{\tau_s(\sigma)}$ is the kernel of the integral operator

$$e^{-t\left(\widetilde{D}(\sigma)\right)^2} f(g) = e^{-tc(\sigma)} \int_G H_t^{\tau_s(\sigma)}\left(g^{-1}g'\right) f\left(g'\right) \mathrm{d}g',$$

where $(\widetilde{D}(\sigma))^2$ is as in (6.7). $H_t^{\tau_s(\sigma)}$ belongs to the Harish-Chandra $L^q$-Schwartz space $(\mathcal{C}^q(G) \otimes \mathrm{End}(V_{\tau_s(\sigma)}))^{K\times K}$. In [24, p. 77], it is proved that $\widetilde{D}_{\tau_s(\sigma)}(\pi)$ maps $(\mathcal{H}_\pi^\infty \otimes V_{\tau_s(\sigma)})^K$ to $(\mathcal{H}_\pi^\infty \otimes V_{\tau_s(\sigma)})^K$. By (6.11) we have

$$\widetilde{\pi}\left(K_t^{\tau_s(\sigma)}\right) = e^{-tc(\sigma)} \widetilde{D}_{\tau_s(\sigma)}(\pi) \circ \widetilde{\pi}\left(H_t^{\tau_s(\sigma)}\right). \tag{6.14}$$

The operator $\widetilde{\pi}(H_t^{\tau_s(\sigma)})$, relative to the splitting

$$\mathcal{H}_\pi \otimes V_{\tau_s(\sigma)} = (\mathcal{H}_\pi \otimes V_{\tau_s(\sigma)})^K \oplus [(\mathcal{H}_\pi \otimes V_{\tau_s(\sigma)})^K]^\perp,$$

takes the form

$$\widetilde{\pi}\left(H_t^{\tau_s(\sigma)}\right) = \begin{pmatrix} \pi\left(H_t^{\tau_s(\sigma)}\right) & 0 \\ 0 & 0 \end{pmatrix}, \tag{6.15}$$

with $\pi(H_t^{\tau_s(\sigma)})$ acting on $(\mathcal{H}_\pi \otimes V_{\tau_s(\sigma)})^K$. Then, by [1, Corollary 2.2], it follows that

$$e^{t\pi(\Omega)}\,\mathrm{Id} = \pi\left(H_t^{\tau_s(\sigma)}\right), \tag{6.16}$$

where Id denotes the identity on the space $(\mathcal{H}_\pi^\infty \otimes V_{\tau_s(\sigma)})^K$. Here, the term $e^{-\lambda_\tau t}$ does nor occur, since the operator $\widetilde{D}(\sigma)^2$ is induced by $\widetilde{A}_\tau$ [see (6.7)]. We have $(\mathcal{H}_\pi \otimes V_{\tau_s(\sigma)})^K = (\mathcal{H}_\pi^\infty \otimes V_{\tau_s(\sigma)})^K$ and

$$\mathrm{Tr}\left(\pi\left(k_t^{\tau_s(\sigma)}\right)\right) = e^{(\pi(\Omega)-c(\sigma))t}\,\mathrm{Tr}\left(\widetilde{D}_{\tau_s(\sigma)}(\pi)|_{(\mathcal{H}_\pi^\infty \otimes V_{\tau_s(\sigma)})^K}\right). \tag{6.17}$$

We recall that the representation space of $\tau_s(\sigma)$ is given by $V_{\tau_s(\sigma)} = V_{\tau(\sigma)} \otimes S$. Let $\pi$ be the unitary principal series representation $\pi_{\sigma,\lambda}$ defined as in Sect. 2. By [22, Proposition 3.6] we have for $(\sigma', V_{\sigma'}) \in \widehat{M}$,

$$\mathrm{Tr}\left(\widetilde{D}_{\tau_s(\sigma)}\left(\pi_{\sigma',\lambda}\right)\right) = \lambda\big(\dim(V_{\sigma'} \otimes V_{\tau(\sigma)} \otimes S^+)^M - \dim(V_{\sigma'} \otimes V_{\tau(\sigma)} \otimes S^-)^M\big). \tag{6.18}$$

Following [24, Corollary 7.6] we let $\check{\sigma}'$ be the contragredient representation of $\sigma'$. Since $\check{\sigma}' \cong \sigma'$, we observe by Eq. (5.17) in Proposition 5.4 that for $\nu_n > 0$,

$$\left[\dim\left(V_{\sigma'} \otimes V_{\tau(\sigma)} \otimes S^+\right)^M - \dim\left(V_{\sigma'} \otimes V_{\tau(\sigma)} \otimes S^-\right)^M\right] = [\sigma - w\sigma : \sigma']. \tag{6.19}$$

Since $\sigma' \in \widehat{M}$ we have that $\sigma' \in \{\sigma, w\sigma\}$, otherwise the right-hand side of (6.19) vanishes. If we put together (5.36), (6.17), (6.18) and (6.19) we obtain

$$\Theta_{\sigma',\lambda}\left(k_t^{\tau_s(\sigma)}\right) = \lambda e^{-t\lambda^2}, \quad \text{if } \sigma' = \sigma \tag{6.20}$$

$$\Theta_{\sigma',\lambda}\left(k_t^{\tau_s(\sigma)}\right) = -\lambda e^{-t\lambda^2}, \quad \text{if } \sigma' = w\sigma \tag{6.21}$$

$$\Theta_{\sigma',\lambda}\left(k_t^{\tau_s(\sigma)}\right) = 0, \quad \text{if } \sigma' \notin \{\sigma, w\sigma\}. \tag{6.22}$$

For the identity contribution we use the fact that when $s$ is restricted to $M$ it decomposes as $s^+ + s^-$. Furthermore, $s^+$ and $s^-$ are connected by the relation $s^- = ws^+$ [2, p. 20]. The Plancherel polynomial is an even polynomial of $\lambda$ and also $P_{s^+}(i\lambda) = P_{ws^+}(-i\lambda) = P_{s^-}(i\lambda)$. Hence,

$$
\begin{aligned}
k_t^{\tau_s(\sigma)}(e) &= \sum_{\sigma\in\widehat{M}} \int_{\mathbb{R}} \Theta_{\sigma,\lambda}(k_t^{\tau_s(\sigma)}) P_\sigma(i\lambda)\mathrm{d}\lambda \\
&= \int_{\mathbb{R}} \lambda e^{-t\lambda^2} P_{s^+}(i\lambda) d\lambda + \int_{\mathbb{R}} -\lambda e^{-t\lambda^2} P_{s^-}(i\lambda)\mathrm{d}\lambda = 0.
\end{aligned}
\tag{6.23}
$$

For the hyperbolic contribution we use (6.20)–(6.22) to get

$$
\mathrm{Tr}\left(D_\chi^\sharp(\sigma)e^{-tD_\chi^\sharp(\sigma)^2}\right) = \frac{1}{2\pi}\sum_{[\gamma]\neq e} \frac{l(\gamma)\,\mathrm{tr}(\chi(\gamma)\otimes(\sigma(m_\gamma)-w\sigma(m_\gamma)))}{D(\gamma)n_\Gamma(\gamma)} \int_{\mathbb{R}} \lambda e^{-t\lambda^2} e^{-il(\gamma)\lambda}\mathrm{d}\lambda.
$$

Equivalently,

$$
\mathrm{Tr}\left(D_\chi^\sharp(\sigma)e^{-tD_\chi^\sharp(\sigma)^2}\right) = \sum_{[\gamma]\neq e} \frac{-2\pi i}{(4\pi t)^{3/2}} \frac{l^2(\gamma)\,\mathrm{tr}(\chi(\gamma)\otimes(\sigma(m_\gamma)-w\sigma(m_\gamma)))}{n_\Gamma(\gamma)D(\gamma)} e^{-l^2(\gamma)/4t}.
$$

All in all, we have proved the following theorem.

**Theorem 6.1** *For every $\sigma \in \widehat{M}$,*

$$
\mathrm{Tr}(D_\chi^\sharp(\sigma)e^{-tD_\chi^\sharp(\sigma)^2}) = \sum_{[\gamma]\neq e} \frac{-2\pi i}{(4\pi t)^{3/2}} \frac{l^2(\gamma)\,\mathrm{tr}(\chi(\gamma)\otimes(\sigma(m_\gamma)-w\sigma(m_\gamma)))}{n_\Gamma(\gamma)D(\gamma)} e^{-l^2(\gamma)/4t}.
\tag{6.24}
$$

# 7 Meromorphic continuation of the zeta functions

## 7.1 Resolvent identities

Let $A$ be a closed linear operator, defined on a dense subspace $\mathcal{D}(A)$ of a Hilbert space $\mathcal{H}$. Let $a \in \mathbb{C} - \mathrm{spec}(-A)$. We set $R(a) := (A + a\,\mathrm{Id})^{-1} = (A + a)^{-1}$. Then, the resolvent identity states

$$
R(a) - R(b) = (b - a)R(a)R(b),
$$

for all $a, b \in \mathbb{C} - \mathrm{spec}(-A)$. The generalized resolvent identity is described in the following Lemma.

**Lemma 7.1** *Let $s_1, \ldots, s_N \in \mathbb{C} - \mathrm{spec}(-A)$, $N \in \mathbb{N}$, such that $s_i \neq s_j$ for all $i \neq j$. Then,*

$$
\prod_{i=1}^N R(s_i) = \sum_{i=1}^N \left( \prod_{\substack{j=1\\ j\neq i}}^N \frac{1}{s_j - s_i} \right) R(s_i).
\tag{7.1}
$$

*Proof* This is proved in [2, Lemma 3.5]. □

**Lemma 7.2** *For $N \in \mathbb{N}$, let $s_1, \ldots, s_N \in \mathbb{C}$ such that $s_i \neq s_j$ for all $i \neq j$ and $l = 0, 1, \ldots, N-2$. Then, we have*

$$\sum_{i=1}^{N} s_i^l \left( \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{1}{s_j - s_i} \right) = 0. \tag{7.2}$$

*Proof* See [2, Lemma 3.6]. □

**Lemma 7.3** *For $N \in \mathbb{N}$, let $s_1, \ldots, s_N \in \mathbb{C}$, such that $s_i^2 \neq s_j^2$ for all $i \neq j$. Then,*

$$\sum_{i=1}^{N} \left( \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} = O(t^{N-1}), \tag{7.3}$$

*as $t \to 0^+$.*

*Proof* We will use the Taylor expansion of the exponential function $e^{-ts_i^2}$. We have as $t \to 0^+$

$$\begin{aligned} \sum_{i=1}^{N} \left( \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} &= \sum_{k=1}^{N-2} \sum_{i=1}^{N} \frac{(-t)^k}{k!} s_i^{2k} \left( \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{1}{s_j^2 - s_i^2} \right) + O(t^{N-1}) \\ &= \sum_{k=1}^{N-2} \frac{(-t)^k}{k!} \sum_{i=1}^{N} s_i^{2k} \left( \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{1}{s_j^2 - s_i^2} \right) + O(t^{N-1}) = O(t^{N-1}), \end{aligned}$$

where in the last equality we used Lemma 7.2. □

**Lemma 7.4** *Let $N \in \mathbb{N}$ with $N > d/2$. Let $s_i \in \mathbb{C}$ such that $\mathrm{Re}(s_i^2) > 0$ for all $i = 1, \ldots, N$. Then, the following integral*

$$\int_0^\infty \int_{\mathbb{R}} \sum_{k=1}^{N} \left( \prod_{\substack{j=1 \\ j \neq k}}^{N} \frac{1}{s_j^2 - s_k^2} \right) e^{-t\left(s_k^2 + \lambda^2\right)} P_\sigma(i\lambda) \mathrm{d}\lambda \mathrm{d}t \tag{7.4}$$

*converges absolutely.*

*Proof* We have as $t \to \infty$,

$$\int_{\mathbb{R}} \left| \sum_{k=1}^{N} \left( \prod_{\substack{j=1 \\ j \neq k}}^{N} \frac{1}{s_j^2 - s_k^2} \right) e^{-t\left(s_k^2 + \lambda^2\right)} P_\sigma(i\lambda) \right| \mathrm{d}\lambda = O(e^{-t\epsilon}), \tag{7.5}$$

for some $\epsilon > 0$. We use now the fact the $P(i\lambda)$ is an even polynomial of degree $2n$ [19, pp. 264–265]. If we make a change of variables $\lambda = \lambda'/\sqrt{t}$, we get as $t \to 0^+$,

$$\int_{\mathbb{R}} \left| e^{-t\lambda^2} P_\sigma(i\lambda) \right| \mathrm{d}\lambda = O(t^{-d/2}). \tag{7.6}$$

Hence, if we combine (7.3) and (7.6) we have that as $t \to 0^+$,

$$\int_{\mathbb{R}} \left| \sum_{k=1}^{N} \left( \prod_{\substack{j=1 \\ j \neq k}}^{N} \frac{1}{s_j^2 - s_k^2} \right) e^{-t(s_k^2+\lambda^2)} P_\sigma(i\lambda) \right| \mathrm{d}\lambda = O(t^{-d/2+N-1}). \tag{7.7}$$

The assertion follows from (7.5) and (7.7). □

### 7.2 Meromorphic continuation of the super zeta function

Let $N \in \mathbb{N}$. Let $s_i$, $i = 1, \ldots, N$ be complex numbers such that $s_i^2 \in \mathbb{C} - \operatorname{spec}(-D_\chi^\sharp(\sigma)^2)$. We consider the resolvent operator $R(s_i^2) = (D_\chi^\sharp(\sigma)^2 + s_i^2)^{-1}$. We want to choose $N$ so large such that the operators $\prod_{i=1}^N R(s_i^2)$ and $D_\chi^\sharp(\sigma) \prod_{i=1}^N R(s_i^2)$ are trace class operators.

**Lemma 7.5** *Let $N \in \mathbb{N}$. Then,*

1. *for $N > \frac{d}{2}$, the operator $\prod_{i=1}^N R(s_i^2)$ is a trace class operator.*
2. *for $N > \frac{d}{2} + 1$, the operator $D_\chi^\sharp(\sigma) \prod_{i=1}^N R(s_i^2)$ is a trace class operator.*

*Proof* We denote the space of pseudodifferential operators of order $k$ by $\psi DO^k$. To prove the trace class property of the operators, we observe at first that $\prod_{i=1}^N R(s_i^2) \in \psi DO^{-2N}$ and $D_\chi^\sharp(\sigma) \prod_{i=1}^N R(s_i^2) \in \psi DO^{-2N+1}$. Choose a metric in $E_\chi$. Let $D$, $\Delta$ be the Dirac and Bochner–Laplace operator, respectively, with respect to the metric in $E_{\tau_s(\sigma)} \otimes E_\chi$, acting on $C^\infty(X, E_{\tau_s(\sigma)} \otimes E_\chi)$. Then, $\Delta$ is a second-order elliptic differential operator, which is formally self-adjoint and nonnegative, i.e., $\Delta \geq 0$.

1. By Weyl's law, we have that for $N > \frac{d}{2}$, $(\Delta + \mathrm{Id})^{-N}$ is a trace class operator. Moreover, $A := (\Delta + \mathrm{Id})^N \prod_{i=1}^N R(s_i^2)$ is $\psi DO$ of order zero. Hence, it defines a bounded operator in $L^2(X, E_{\tau_s(\sigma)} \otimes E_\chi)$. Thus, $\prod_{i=1}^N R(s_i^2) = (\Delta + \mathrm{Id})^{-N} A$ is a trace class operator.
2. Similarly, by Weyl's law, we have that for $N > \frac{d}{2} + 1$, $(\Delta + \mathrm{Id})^{-N} D$ is a trace class operator. We define $B := D^{-1}(\Delta + \mathrm{Id})^N D_\chi^\sharp(\sigma) \prod_{i=1}^N R(s_i^2)$. Then, $B$ is $\psi DO$ of order zero and defines a bounded operator in $L^2(X, E_{\tau_s(\sigma)} \otimes E_\chi)$. Hence, $D_\chi^\sharp(\sigma) \prod_{i=1}^N R(s_i^2) = (\Delta + \mathrm{Id})^{-N} DB$ is a trace class operator. □

We recall here the following expressions of the resolvents. Let $s_1, \ldots, s_N \in \mathbb{C}$ such that $\operatorname{Re}(s_i^2) > -c$, for all $i = 1, \ldots, N$, where $c$ is a real number such that $\operatorname{spec}\left(D_\chi^\sharp(\sigma)^2\right) \subset \{z \in \mathbb{C} \colon \operatorname{Re}(z) > c\}$. Then,

$$D_\chi^\sharp(\sigma)(D_\chi^\sharp(\sigma)^2 + s_i^2)^{-1} = \int_0^\infty e^{-ts_i^2} D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2} \mathrm{d}t \tag{7.8}$$

$$\left(A_\chi^\sharp(\sigma) + s_i^2\right)^{-1} = \int_0^\infty e^{-ts_i^2} e^{-tA_\chi^\sharp(\sigma)} \mathrm{d}t. \tag{7.9}$$

**Lemma 7.6** *Let $\| D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2} \|_1$ be the trace norm of the operator $D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2}$. Then, there exists a positive constant $c$ such that for $0 < t \leq 1$,*

$$\left\| D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2} \right\|_1 \leq ct^{-1-d}.$$

*Proof* We have

$$\left\| D^{\sharp}_{\chi}(\sigma)e^{-tD^{\sharp}_{\chi}(\sigma)^2} \right\|_1 \le \left\| D^{\sharp}_{\chi}(\sigma)e^{-t/2D^{\sharp}_{\chi}(\sigma)^2} \right\| \left\| e^{-t/2D^{\sharp}_{\chi}(\sigma)^2} \right\|_1. \tag{7.10}$$

Let $\Gamma_{\theta,r_0}$ be defined as in "Appendix" (pp. 54–55). For $\lambda \in \Gamma_{\theta,r_0}$ and every $t > 0$, we have $|t\lambda e^{-(t/2)\lambda}| \le M$, where $M > 0$. In addition, by [26, Theorem 9.3], there exists a positive constant $c_0$ such that for $\lambda \in \Gamma_{\theta,r_0}$,

$$\left\| D^{\sharp}_{\chi}(\sigma)\big(D^{\sharp}_{\chi}(\sigma)^2 - \lambda\, \mathrm{Id}\big)^{-1} \right\| \le \left\| \big(D^{\sharp}_{\chi}(\sigma)^2 - \lambda\, \mathrm{Id}\big)^{-1} \right\|_{H^1(X)} \le c_0 |\lambda|^{-1/2}.$$

Then,

$$\begin{aligned} \int_{\Gamma_{\theta,r_0}} |te^{-(t/2)\lambda}| \left\| D^{\sharp}_{\chi}(\sigma)\big(D^{\sharp}_{\chi}(\sigma)^2 - \lambda\, \mathrm{Id}\big)^{-1} \right\| \mathrm{d}\lambda &\le c_0 \int_{\Gamma_{\theta,r_0}} |te^{-(t/2)\lambda}||\lambda|^{-1/2} \mathrm{d}\lambda \\ &\le c_0 \int_{\Gamma_{\theta,r_0}} M|\lambda|^{-3/2} \mathrm{d}\lambda = c_1, \end{aligned}$$

where $c_1$ is a positive constant. Hence, by (8.1) (see "Appendix") we get

$$\begin{aligned} \left\| D^{\sharp}_{\chi}(\sigma)e^{-t/2D^{\sharp}_{\chi}(\sigma)^2} \right\| &= \left\| D^{\sharp}_{\chi}(\sigma) \left( \frac{i}{2\pi} \int_{\Gamma_{\theta,r_0}} e^{-(t/2)\lambda} \big(D^{\sharp}_{\chi}(\sigma)^2 - \lambda\, \mathrm{Id}\big)^{-1} \mathrm{d}\lambda \right) \right\| \\ &\le \frac{1}{t} \int_{\Gamma_{\theta,r_0}} |te^{-(t/2)\lambda}| \left\| D^{\sharp}_{\chi}(\sigma)\big(D^{\sharp}_{\chi}(\sigma)^2 - \lambda\, \mathrm{Id}\big)^{-1} \right\| \mathrm{d}\lambda \le c_1 \frac{1}{t}. \end{aligned} \tag{7.11}$$

Let $\|\cdot\|_2$ denote the Hilbert-Schmidt norm. Then,

$$\left\| e^{-t/2D^{\sharp}_{\chi}(\sigma)^2} \right\|_1 = \left\| e^{-t/4D^{\sharp}_{\chi}(\sigma)^2} e^{-t/4D^{\sharp}_{\chi}(\sigma)^2} \right\|_1 \le \left\| e^{-t/4D^{\sharp}_{\chi}(\sigma)^2} \right\|_2 \left\| e^{-t/4D^{\sharp}_{\chi}(\sigma)^2} \right\|_2. \tag{7.12}$$

Moreover,

$$\left\| e^{-t/4D^{\sharp}_{\chi}(\sigma)^2} \right\|_2^2 = \mathrm{Tr}\left( e^{-t/4(D^{\sharp}_{\chi}(\sigma)^2)^*} e^{-t/4D^{\sharp}_{\chi}(\sigma)^2} \right). \tag{7.13}$$

Since $\frac{\partial}{\partial t}\mathrm{Tr}(e^{-t/4(D^{\sharp}_{\chi}(\sigma)^2)^*} e^{-t/4D^{\sharp}_{\chi}(\sigma)^2}) = \frac{\partial}{\partial t}\mathrm{Tr}(e^{-t/4((D^{\sharp}_{\chi}(\sigma)^2)^* + D^{\sharp}_{\chi}(\sigma)^2)})$, the traces only differ by a constant. By [9, Lemma 1.8.2], we have

$$\mathrm{Tr}\left( e^{-t/4\left(\left(D^{\sharp}_{\chi}(\sigma)^2\right)^* + D^{\sharp}_{\chi}(\sigma)^2\right)} \right) \sim_{t\to 0^+} \dim(V_\chi) \sum_{j=0}^{\infty} a_j t^{j-d/2},$$

where the coefficients $a_j$ are determined by smooth local invariants on $X$. Hence, for $0 < t \le 1$,

$$\left|\mathrm{Tr}\left( e^{-t/4\left(\left(D^{\sharp}_{\chi}(\sigma)^2\right)^* + D^{\sharp}_{\chi}(\sigma)^2\right)} \right)\right| \le c_2 t^{-d/2}, \tag{7.14}$$

where $c_2$ is a positive constant. By (7.13), (7.14) and the remark above, (7.12) becomes

$$\left\| e^{-t/2D^{\sharp}_{\chi}(\sigma)^2} \right\|_1 \le c_3 t^{-d}, \tag{7.15}$$

where $c_3$ is a positive constant. Therefore, by (7.11) and (7.15), (7.10) becomes

$$\left\| D^{\sharp}_{\chi}(\sigma)e^{-tD^{\sharp}_{\chi}(\sigma)^2} \right\|_1 \leq c_4 t^{-1-d},$$

where $c_4$ is a positive constant. □

**Proposition 7.7** *Let $N \in \mathbb{N}$ with $N > d/2+1$. Let $s_1, \ldots, s_N \in \mathbb{C}$ with $s_i^2 \neq s_j^2$ for all $i \neq j$ such that $\mathrm{Re}(s_i^2) > -c$, for all $i = 1, \ldots, N$, where $c$ is a real number such that $\mathrm{spec}\left(D^{\sharp}_{\chi}(\sigma)^2\right) \subset \{z \in \mathbb{C} \colon \mathrm{Re}(z) > c\}$. Let $L^s(s) := \frac{d}{ds}\log(Z^s(s;\sigma,\chi))$ be the logarithmic derivative of the super zeta function. Then,*

$$\mathrm{Tr}\left(D^{\sharp}_{\chi}(\sigma)\prod_{i=1}^{N}\left(D^{\sharp}_{\chi}(\sigma)^2 + s_i^2\right)^{-1}\right) = -\frac{i}{2}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2 - s_i^2}\right)L^s(s_i). \tag{7.16}$$

*Proof* By Lemma 7.1 and Eq. (7.8), we have

$$D^{\sharp}_{\chi}(\sigma)\prod_{i=1}^{N}\left(D^{\sharp}_{\chi}(\sigma)^2 + s_i^2\right)^{-1} = \int_0^{\infty}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2 - s_i^2}\right)e^{-ts_i^2}D^{\sharp}_{\chi}(\sigma)e^{-tD^{\sharp}_{\chi}(\sigma)^2}\mathrm{d}t.$$

The operators $D^{\sharp}_{\chi}(\sigma)\prod_{i=1}^{N}(D^{\sharp}_{\chi}(\sigma)^2 + s_i^2)^{-1}$ and $D^{\sharp}_{\chi}(\sigma)e^{-tD^{\sharp}_{\chi}(\sigma)^2}$ are both of trace class. We consider the trace:

$$\mathrm{Tr}\left(\int_0^{\infty}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2 - s_i^2}\right)e^{-ts_i^2}D^{\sharp}_{\chi}(\sigma)e^{-tD^{\sharp}_{\chi}(\sigma)^2}\mathrm{d}t\right).$$

For $0 < \epsilon < R$, we decompose the integral above into the sum of integrals over $[0,\epsilon]$, $[\epsilon, R]$, and $[R,\infty)$. Since $e^{-ts_i^2}$ and $\mathrm{Tr}(D^{\sharp}_{\chi}(\sigma)e^{-tD^{\sharp}_{\chi}(\sigma)^2})$ decay exponentially, for $R$ sufficiently large, the integral over $[R,\infty)$ can be made arbitrarily small. Moreover, by Lemma 7.6, there exists a positive constant $c$ such that for $0 < t \leq 1$, $\|D^{\sharp}_{\chi}(\sigma)e^{-tD^{\sharp}_{\chi}(\sigma)^2}\|_1 \leq ct^{-1-d}$. Using Lemma 7.3, it follows that for $\epsilon > 0$ sufficiently small, the integral over $[0,\epsilon]$ can be made arbitrarily small.

Therefore,

$$\begin{aligned}
&\mathrm{Tr}\left(\int_0^{\infty}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2 - s_i^2}\right)e^{-ts_i^2}D^{\sharp}_{\chi}(\sigma)e^{-tD^{\sharp}_{\chi}(\sigma)^2}\mathrm{d}t\right)\\
&= \lim_{\substack{\epsilon\to 0\\ R\to\infty}}\mathrm{Tr}\left(\int_{\epsilon}^{R}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2 - s_i^2}\right)e^{-ts_i^2}D^{\sharp}_{\chi}(\sigma)e^{-tD^{\sharp}_{\chi}(\sigma)^2}\mathrm{d}t\right)\\
&= \lim_{\substack{\epsilon\to 0\\ R\to\infty}}\int_{\epsilon}^{R}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2 - s_i^2}\right)e^{-ts_i^2}\,\mathrm{Tr}(D^{\sharp}_{\chi}(\sigma)e^{-tD^{\sharp}_{\chi}(\sigma)^2})\mathrm{d}t, \qquad (7.17)
\end{aligned}$$

where in the second equality in the equation above we used the fact that the trace norm $\|D^{\sharp}_{\chi}(\sigma)e^{-tD^{\sharp}_{\chi}(\sigma)^2}\|_1$ is continuous on $[\epsilon, R]$, and therefore, summation and integration can be interchanged.

As $R \to \infty$, $e^{-ts_i^2}$ and $\mathrm{Tr}(D^{\sharp}_{\chi}(\sigma)e^{-tD^{\sharp}_{\chi}(\sigma)^2})$ in the last integral decay exponentially. We study now the behavior of this integral as $\epsilon \to 0$.

By [28, Lemma 3.1], there exists a short-time asymptotic expansion of the kernel of the operator $D^{\sharp}_{\chi}(\sigma)e^{-tD^{\sharp}_{\chi}(\sigma)^2}$. We have

$$\mathrm{Tr}\left(D^{\sharp}_{\chi}(\sigma)e^{-tD^{\sharp}_{\chi}(\sigma)^2}\right) \sim_{t\to 0^+} \dim(V_{\chi})(a_0 t^{1/2} + O(t^{3/2})),$$

where $a_0$ is determined by a smooth local invariant on $X$ (see [28][pp. 16–17]). Then, by Lemma 7.3, for $0 < t \leq 1$,

$$\left|\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2 - s_i^2}\right)e^{-ts_i^2}\,\mathrm{Tr}\left(D^{\sharp}_{\chi}(\sigma)e^{-tD^{\sharp}_{\chi}(\sigma)^2}\right)\right| \leq Ct^{1/2},$$

where $C$ is a positive constant. Hence, the limit of the integral on the right-hand side of (7.17) exists and we get

$$\begin{aligned}\mathrm{Tr}\left(D^{\sharp}_{\chi}(\sigma)\prod_{i=1}^{N}\left(D^{\sharp}_{\chi}(\sigma)^2 + s_i^2\right)^{-1}\right) &= \mathrm{Tr}\left(\int_0^{\infty}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2 - s_i^2}\right)e^{-ts_i^2}D^{\sharp}_{\chi}(\sigma)e^{-tD^{\sharp}_{\chi}(\sigma)^2}\,\mathrm{d}t\right)\\ &= \int_0^{\infty}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2 - s_i^2}\right)e^{-ts_i^2}\,\mathrm{Tr}\left(D^{\sharp}_{\chi}(\sigma)e^{-tD^{\sharp}_{\chi}(\sigma)^2}\right)\mathrm{d}t.\end{aligned}$$

We insert the right-hand side of the trace formula (6.24) in the last integral of the equation above. Then, we get

$$\begin{aligned}\mathrm{Tr}\left(D^{\sharp}_{\chi}(\sigma)\prod_{i=1}^{N}\left(D^{\sharp}_{\chi}(\sigma)^2 + s_i^2\right)^{-1}\right) &= \int_0^{\infty}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2 - s_i^2}\right)e^{-ts_i^2}\\ &\left\{\sum_{[\gamma]\neq e}\frac{-2\pi i}{(4\pi t)^{3/2}}\frac{l^2(\gamma)\,\mathrm{tr}(\chi(\gamma)\otimes(\sigma(m_\gamma) - w\sigma(m_\gamma)))}{n_\Gamma(\gamma)D(\gamma)}e^{-l^2(\gamma)/4t}\right\}.\end{aligned} \tag{7.18}$$

If we use the formula

$$\int_0^{\infty} e^{-ts^2}\frac{1}{(4\pi t)^{3/2}}e^{-l^2(\gamma)/4t}\mathrm{d}t = \frac{e^{-l(\gamma)s}}{4\pi l(\gamma)}$$

(see [5, p. 146, (28)]), Eq. (7.18) becomes

$$\operatorname{Tr}\left(D^{\sharp}_{\chi}(\sigma)\prod_{i=1}^{N}\left(D^{\sharp}_{\chi}(\sigma)^{2}+s_i^2\right)^{-1}\right)=\frac{-i}{2}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\left\{\sum_{[\gamma]\neq e}\frac{l(\gamma)\operatorname{tr}\left(\chi(\gamma)\otimes(\sigma(m_\gamma)-w\sigma(m_\gamma))\right)}{n_\Gamma(\gamma)D(\gamma)}e^{-l(\gamma)s_i}\right\}.$$

Recall that $L^s(s_i)$ is the logarithmic derivative of the super zeta function $Z^s(s_i;\sigma,\chi)$. By (3.25) we get

$$\operatorname{Tr}\left(D^{\sharp}_{\chi}(\sigma)\prod_{i=1}^{N}\left(D^{\sharp}_{\chi}(\sigma)^{2}+s_i^2\right)^{-1}\right)=\frac{-i}{2}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)L^s(s_i).$$

□

**Theorem 7.8** *The super zeta function* $Z^s(s;\sigma,\chi)$ *admits a meromorphic continuation to the whole complex plane* $\mathbb{C}$. *The singularities are located at* $\{s_k^{\pm}=\pm i\lambda_k:\lambda_k\in\operatorname{spec}(D^{\sharp}_{\chi}(\sigma)),k\in\mathbb{N}\}$ *of order* $\pm m_s(\lambda_k)$, *where* $m_s(\lambda_k)=m(\lambda_k)-m(-\lambda_k)\in\mathbb{N}$ *and* $m(\pm\lambda_k)$ *denotes the algebraic multiplicity of the eigenvalue* $\pm\lambda_k$.

*Proof* We define the function $\Phi(s_1,s_2,\ldots,s_N)$ of the complex variables $s_1,s_2,\ldots,s_N$ by

$$\Phi(s_1,s_2,\ldots,s_N)=-\frac{i}{2}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)L^s(s_i). \tag{7.19}$$

By Lidskii's theorem and Lemma 7.1, (7.16) becomes

$$\sum_{\lambda_k}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)m_s(\lambda_k)\lambda_k\frac{1}{(\lambda_k)^2+s_i^2}=\Phi(s_1,s_2,\ldots,s_N). \tag{7.20}$$

We fix the complex numbers $s_i$, $i=2,\ldots,N$ with $s_i\neq s_j$ for $i,j=2,\ldots,N$ and let the complex number $s=s_1$ vary. Hence,

$$\Phi(s,s_2,\ldots,s_N)=\Phi(s)$$

The term that contains the logarithmic derivative $L^s(s)$ in $\Phi(s)$ is of the form

$$\frac{-i}{2}\left(\prod_{j=2}^{N}\frac{1}{s_j^2-s^2}\right)L^s(s). \tag{7.21}$$

We show now that $L^s(s)$ is a meromorphic function of $s$. By (7.16), $\Phi(s,s_2,\ldots,s_N)$ is a meromorphic function of $s$, since the resolvent operator $(D^{\sharp}_{\chi}(\sigma)^2+s^2)^{-1}$ is a meromorphic function of $s$. The term of

$$\sum_{\lambda_k}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)m_s(\lambda_k)\lambda_k\frac{1}{(\lambda_k)^2+s_i^2}$$

which is singular at $s=\pm i\lambda_k$, $k\in\mathbb{N}$, is

$$\left(\prod_{j=2}^{N}\frac{1}{s_j^2-s^2}\right)m_s(\lambda_k)\lambda_k\frac{1}{(\lambda_k)^2+s^2}.$$

If we multiply both sides of (7.20) by

$$2i\prod_{j=2}^{N}(s_j^2-s^2),$$

we see that the residue of the logarithmic derivative $L^s(s)$ at $\pm i\lambda_k$ is $\pm m_s(\lambda_k)$. By (3.25), $L^s(s)$ decreases exponentially as $\mathrm{Re}(s)\to\infty$. Hence, the integral

$$\int_s^\infty L^s(w)\mathrm{d}w$$

over a path connecting $s$ and infinity is well defined and

$$\log Z^s(s;\sigma,\chi)=-\int_s^\infty L^s(w)\mathrm{d}w. \tag{7.22}$$

The integral above depends on the choice of the path, because $L^s(s)$ has singularities at $s_k^{\pm}$. Nevertheless, since all the residues of the singularities are integers, it follows that the exponential of the integral in the right-hand side of (7.22) is independent of the choice of the path. The meromorphic continuation of the super zeta function $Z^s(s;\sigma,\chi)$ to the whole complex plane follows. □

### 7.3 Meromorphic continuation of the Selberg zeta function

We study first the case where $\sigma\in\widehat{M}$ is Weyl invariant. Let $N\in\mathbb{N}$ with $N>d/2$. Let $s_1,\ldots,s_N\in\mathbb{C}$ with $s_i^2\neq s_j^2$ for all $i\neq j$ such that $\mathrm{Re}(s_i^2)>-C$, for all $i=1,\ldots,N$, where $C$ is a real number such that $\operatorname{spec}\left(A_\chi^\sharp(\sigma)\right)\subset\{z\in\mathbb{C}\colon \mathrm{Re}(z)>C\}$. By Lemma 7.1 and Eq. (7.9) we have

$$\prod_{i=1}^{N}\left(A_\chi^\sharp(\sigma)+s_i^2\right)^{-1}=\int_0^\infty\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)e^{-ts_i^2}e^{-tA_\chi^\sharp(\sigma)}\mathrm{d}t.$$

Arguing as in the proof of Proposition 7.7, we can plug the trace in both sides of the equation above. Then,

$$\mathrm{Tr}\prod_{i=1}^{N}\left(A_\chi^\sharp(\sigma)+s_i^2\right)^{-1}=\int_0^\infty\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)e^{-ts_i^2}\,\mathrm{Tr}\,e^{-tA_\chi^\sharp(\sigma)}\mathrm{d}t.$$

We insert the right-hand side of the trace formula (5.38) in the right-hand side of the equation above.

$$\mathrm{Tr}\prod_{i=1}^{N}\left(A^{\sharp}_{\chi}(\sigma)+s_i^2\right)^{-1}=\int_0^\infty\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)e^{-ts_i^2}$$

$$\left\{\dim(V_\chi)\,\mathrm{Vol}(X)\int_{\mathbb{R}}e^{-t\lambda^2}P_\sigma(i\lambda)\mathrm{d}\lambda+\sum_{[\gamma]\neq e}\frac{l(\gamma)}{n_\Gamma(\gamma)}L(\gamma;\sigma,\chi)\frac{e^{-l(\gamma)^2/(4t)}}{(4\pi t)^{1/2}}\right\}\mathrm{d}t.$$

Hence,

$$\mathrm{Tr}\prod_{i=1}^{N}\left(A^{\sharp}_{\chi}(\sigma)+s_i^2\right)^{-1}=\dim(V_\chi)\,\mathrm{Vol}(X)\int_0^\infty\int_{\mathbb{R}}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)e^{-ts_i^2}e^{-t\lambda^2}P_\sigma(i\lambda)\mathrm{d}\lambda\mathrm{d}t$$

$$+\sum_{i=1}^{N}\int_0^\infty\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)e^{-ts_i^2}\left\{\sum_{[\gamma]\neq[e]}\frac{l(\gamma)}{n_\Gamma(\gamma)}L(\gamma;\sigma,\chi)\frac{e^{-l(\gamma)^2/(4t)}}{(4\pi t)^{1/2}}\right\}\mathrm{d}t. \tag{7.23}$$

The first sum in (7.23), which involves the double integral can be explicitly calculated. We set

$$I:=\int_0^\infty\int_{\mathbb{R}}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)e^{-ts_i^2}e^{-t\lambda^2}P_\sigma(i\lambda)\mathrm{d}\lambda\mathrm{d}t.$$

By Lemma 7.4, we can interchange the order of the integration and get

$$I=\int_{\mathbb{R}}\int_0^\infty\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)e^{-t(s_i^2+\lambda^2)}P_\sigma(i\lambda)\mathrm{d}t\mathrm{d}\lambda.$$

Equivalently,

$$I=\int_{\mathbb{R}}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\left(\frac{1}{\lambda^2+s_i^2}\right)P_\sigma(i\lambda)\mathrm{d}\lambda.$$

Using the residue theorem [32, pp. 127–128] and the fact that $P_\sigma$ is an even polynomial, we have

$$I=\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\frac{\pi}{s_i}P_\sigma(s_i). \tag{7.24}$$

For the second sum in (7.23) we use the formula

$$\int_0^\infty e^{-ts^2}\frac{e^{-l(\gamma)^2/(4t)}}{(4\pi t)^{1/2}}\mathrm{d}t=\frac{1}{2s}e^{-sl(\gamma)}, \tag{7.25}$$

(see [5, p. 146, (27)]). Hence, by (7.24) and (7.25), Eq. (7.23) becomes

$$\operatorname{Tr}\prod_{i=1}^{N}(A^{\sharp}_{\chi}(\sigma)+s_i^2)^{-1}=\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\frac{\pi}{s_i}\dim(V_\chi)\operatorname{Vol}(X)P(s_i)$$
$$+\sum_{i=1}^{N}\frac{1}{2s_i}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\sum_{[\gamma]\neq e}\frac{l(\gamma)}{n_\Gamma(\gamma)}L(\gamma;\sigma,\chi)e^{-s_i l(\gamma)}. \tag{7.26}$$

By (3.23) we have that the sum over the conjugacy classes $[\gamma]$ of $\Gamma$ in the right-hand side of (7.26) is equal to the logarithmic derivative $L(s_i)=\frac{\mathrm{d}}{\mathrm{d}s}\log(Z(s_i;\sigma,\chi)$ of the Selberg zeta function. Hence,

$$\operatorname{Tr}\prod_{i=1}^{N}(A^{\sharp}_{\chi}(\sigma)+s_i^2)^{-1}=\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\frac{\pi}{s_i}\dim(V_\chi)\operatorname{Vol}(X)P(s_i)$$
$$+\sum_{i=1}^{N}\frac{1}{2s_i}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)L(s_i). \tag{7.27}$$

We will use the equation above to prove the following theorem. We choose the branch of the square roots of the complex numbers $t_k$, whose real part is positive. In addition, if $t_k$ are negative real numbers, we choose the branch of the square roots, whose imaginary part is positive.

**Theorem 7.9** *The Selberg zeta function $Z(s;\sigma,\chi)$ admits a meromorphic continuation to the whole complex plane $\mathbb{C}$. The set of the singularities equals $\{s_k^{\pm}=\pm i\sqrt{t_k}: t_k\in \operatorname{spec}(A^{\sharp}_{\chi}(\sigma)), k\in\mathbb{N}\}$. The orders of the singularities are equal to $m(t_k)$, where $m(t_k)\in\mathbb{N}$ denotes the algebraic multiplicity of the eigenvalue $t_k$. For $t_0=0$, the order of the singularity $s_0$ is equal to $2m(0)$.*

*Proof* By (7.1) and (7.27) we get

$$\operatorname{Tr}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)(A^{\sharp}_{\chi}(\sigma)+s_i^2)^{-1}=\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\frac{\pi}{s_i}\dim(V_\chi)\operatorname{Vol}(X)P(s_i)$$
$$+\sum_{i=1}^{N}\frac{1}{2s_i}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)L(s_i).$$

Equivalently,

$$\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\frac{1}{2s_i}L(s_i)=-\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\frac{\pi}{s_i}\dim(V_\chi)\,\mathrm{Vol}(X)P(s_i)$$
$$+\sum_{t_k}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\frac{m(t_k)}{t_k+s_i^2}, \tag{7.28}$$

If we multiply Eq. (7.28) by $2s_1$, we obtain

$$\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\frac{s_1}{s_i}L(s_i)=-\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\frac{2s_1}{s_i}\pi\dim(V_\chi)\,\mathrm{Vol}(X)P(s_i)$$
$$+\sum_{t_k}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)2s_1\frac{m(t_k)}{t_k+s_i^2}. \tag{7.29}$$

Let $\Psi(s_1,\ldots,s_N)$ be the function of the complex numbers $s_1,\ldots,s_N$, defined by

$$\Psi(s_1,\ldots,s_N):=\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\frac{s_1}{s_i}L(s_i).$$

We fix the complex numbers $s_i$, $i=2,\ldots,N$ with $s_i\neq s_j$ for $i,j=2,\ldots,N$ and let the complex number $s=s_1$ vary. Put

$$\Psi(s,\ldots,s_N)=\Psi(s).$$

Then, Eq. (7.29) becomes

$$\Psi(s)=-\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\frac{2s}{s_i}\pi\dim(V_\chi)\,\mathrm{Vol}(X)P(s_i)$$
$$+\sum_{t_k}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)2s\frac{m(t_k)}{t_k+s_i^2}, \tag{7.30}$$

where $s_1=s$. $L(s)$ is a meromorphic function of $s$, since $\Psi(s,\ldots,s_N)$ is a meromorphic function of $s$. This is because by (7.27), the resolvent operator $(A^{\sharp}_{\chi}(\sigma)+s^2)^{-1}$ is a meromorphic function of $s$. The term that contains the logarithmic derivative $L(s)$ in $\Psi(s)$ is of the form

$$\left(\prod_{j=2}^{N}\frac{1}{s_j^2-s^2}\right)L(s). \tag{7.31}$$

The term of

$$\sum_{t_k}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)2s\frac{m(t_k)}{t_k+s_i^2},$$

which is singular at $\pm i\sqrt{t_k}$, $k\in\mathbb{N}$ is

$$\left(\prod_{j=2}^{N}\frac{1}{s_j^2-s^2}\right)2s\frac{m(t_k)}{t_k+s^2}.$$

We multiply both sides of (7.30) by

$$\prod_{j=2}^{N}(s_j^2-s^2).$$

Then, the residues of $L(s)$ at the points $\pm i\sqrt{t_k}$ are $m(t_k)$, for $k\neq 0$, and $2m(0)$ for $k=0$. By (3.23), $L(s)$ decreases exponentially as $\mathrm{Re}(s)\to\infty$. Hence, the integral

$$\int_s^{\infty}L(w)\mathrm{d}w$$

over a path connecting $s$ and infinity is well defined and

$$\log Z(s;\sigma,\chi)=-\int_s^{\infty}L(w)\mathrm{d}w. \tag{7.32}$$

The integral above depends on the choice of the path, because $L(s)$ has singularities. Since the residues of the singularities are integers, it follows as in the proof of Theorem 7.8 that the exponential of the integral in the right-hand side of (7.32) is independent of the choice of the path. The meromorphic continuation of the Selberg zeta function $Z(s;\sigma,\chi)$ to the whole complex plane follows. □

### 7.4 Meromorphic continuation of the symmetrized zeta function

We examine now the case where $\sigma$ is non-Weyl invariant. Let $N\in\mathbb{N}$ with $N>d/2$. We choose $s_1,\ldots,s_N\in\mathbb{C}$ with $s_i^2\neq s_j^2$ for all $i\neq j$ such that $\mathrm{Re}(s_i^2)>-r$, for all $i=1,\ldots,N$, where $r$ is a real number such that $\mathrm{spec}\left(A_\chi^\sharp(\sigma)\right)\subset\{z\in\mathbb{C}\colon \mathrm{Re}(z)>r\}$. Then, by Lemma 7.1 and equation (7.9), we have

$$\prod_{i=1}^{N}(A_\chi^\sharp(\sigma)+s_i^2)^{-1}=\int_0^{\infty}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)e^{-ts_i^2}e^{-tA_\chi^\sharp(\sigma)}\mathrm{d}t.$$

As in the proof of Proposition 7.7, we consider the trace of the operators in the formula above and get

$$\mathrm{Tr}\prod_{i=1}^{N}(A_\chi^\sharp(\sigma)+s_i^2)^{-1}=\int_0^{\infty}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)e^{-ts_i^2}\,\mathrm{Tr}\,e^{-tA_\chi^\sharp(\sigma)}\mathrm{d}t.$$

We insert the right-hand side of the trace formula (5.39) in the right-hand side of the equation above. We have

$$\begin{aligned}\operatorname{Tr}\prod_{i=1}^{N}&(A^{\sharp}_{\tau,\chi}(\sigma)+s_i^2)^{-1}\\&=\int_0^{\infty}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)e^{-ts_i^2}\Bigg\{2\dim(V_\chi)\operatorname{Vol}(X)\int_{\mathbb{R}}e^{-t\lambda^2}P_\sigma(i\lambda)\mathrm{d}\lambda\\&\quad+\sum_{[\gamma]\neq e}\frac{l(\gamma)}{n_\Gamma(\gamma)}L(\gamma;\sigma+w\sigma,\chi)\frac{e^{-l(\gamma)^2/4t}}{(4\pi t)^{1/2}}\Bigg\}\mathrm{d}t.\end{aligned}\tag{7.33}$$

The first term in the right-hand side of (7.33) includes the double integral

$$I=\int_0^{\infty}\int_{\mathbb{R}}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)e^{-ts_i^2}e^{-t\lambda^2}P_\sigma(i\lambda)\mathrm{d}\lambda\mathrm{d}t,$$

which has been computed in Sect. 7.3. Equation (7.24) gives

$$I=\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\frac{\pi}{s_i}P_\sigma(s_i).$$

Hence, Eq. (7.33) reads

$$\begin{aligned}\operatorname{Tr}\prod_{i=1}^{N}(A^{\sharp}_{\chi}(\sigma)+s_i^2)^{-1}&=\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\frac{\pi}{s_i}2\dim(V_\chi)\operatorname{Vol}(X)P_\sigma(s_i)\\&\quad+\sum_{i=1}^{N}\frac{1}{2s_i}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\sum_{[\gamma]\neq[e]}\frac{l(\gamma)}{n_\Gamma(\gamma)}L(\gamma;\sigma+w\sigma,\chi)e^{-s_il(\gamma)}.\end{aligned}$$

By (3.24) we can insert the logarithmic derivative $L_S(s)$ of the symmetrized zeta function in the second sum of the right-hand side of the equation above. Then, we get

$$\begin{aligned}\operatorname{Tr}\prod_{i=1}^{N}(A^{\sharp}_{\chi}(\sigma)+s_i^2)^{-1}&=\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\frac{\pi}{s_i}2\dim(V_\chi)\operatorname{Vol}(X)P_\sigma(s_i)\\&\quad+\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\frac{1}{2s_i}L_S(s_i).\end{aligned}\tag{7.34}$$

Equation (7.34) will give the meromorphic continuation of the symmetrized zeta function. We choose the branch of the square roots of the complex numbers $\mu_k$, whose real part is positive. In case that $\mu_k$ are negative real numbers, we choose the branch of the square roots, whose imaginary part is positive.

**Theorem 7.10** *The symmetrized zeta function $S(s;\sigma,\chi)$ admits a meromorphic continuation to the whole complex plane $\mathbb{C}$. The set of the singularities equals $\{s_k^{\pm}=\pm i\sqrt{\mu_k}:\mu_k\in\operatorname{spec}(A^{\sharp}_{\chi}(\sigma)),k\in\mathbb{N}\}$. The orders of the singularities are equal to $m(\mu_k)$, where $m(\mu_k)\in\mathbb{N}$ denotes the algebraic multiplicity of the eigenvalue $\mu_k$. For $\mu_0=0$, the order of the singularity $s_0$ is equal to $2m(0)$.*

*Proof* Lemma (7.1) and Eq. (7.34) give

$$\sum_{\mu_k}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\frac{m(\mu_k)}{\mu_k+s_i^2}=\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\frac{\pi}{s_i}2\dim(V_\chi)\operatorname{Vol}(X)P_\sigma(s_i)$$
$$+\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\frac{1}{2s_i}L_S(s_i).$$

We multiply the last equation by $2s_1$ and get

$$\sum_{\mu_k}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)2s_1\frac{m(\mu_k)}{\mu_k+s_i^2}=\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\frac{4\pi s_1}{s_i}\dim(V_\chi)\operatorname{Vol}(X)P_\sigma(s_i)$$
$$+\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\frac{s_1}{s_i}L_S(s_i). \tag{7.35}$$

We define the function $\Xi(s_1,\dots,s_N)$ of the complex variables $s_1,\dots,s_N$ by

$$\Xi(s_1,\dots,s_N):=\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\frac{s_1}{s_i}L_S(s_i).$$

We fix the complex numbers $s_i, i=2,\dots,N$ with $s_i\neq s_j$ for $i,j=2,\dots,N$ and let the complex number $s=s_1$ vary. Then,

$$\Xi(s,\dots,s_N)=\Xi(s),$$

and Eq. (7.35) becomes

$$\sum_{\mu_k}\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)2s\frac{m(\mu_k)}{\mu_k+s_i^2}=\sum_{i=1}^{N}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\frac{1}{s_j^2-s_i^2}\right)\frac{4\pi s}{s_i}\dim(V_\chi)\operatorname{Vol}(X)P_\sigma(s_i)$$
$$+\Xi(s). \tag{7.36}$$

$L_S(s)$ is a meromorphic function of $s$, since $\Xi(s,\dots,s_N)$ is a meromorphic function of $s$. This follows from (7.34) and the fact that the resolvent operator $(A^{\sharp}_{\chi}(\sigma)+s^2)^{-1}$ is a meromorphic function of $s$. The term that contains the logarithmic derivative $L_S(s)$ in $\Xi(s)$ is of the form

$$\left(\prod_{j=2}^{N} \frac{1}{s_j^2 - s^2}\right) L_S(s). \tag{7.37}$$

The term of

$$\sum_{\mu_k} \sum_{i=1}^{N} \left(\prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{s_j^2 - s_i^2}\right) 2s \frac{m(\mu_k)}{\mu_k + s_i^2},$$

which is singular at $s = \pm i\sqrt{\mu_k}$, $k \in \mathbb{N}$ is

$$\left(\prod_{j=2}^{N} \frac{1}{s_j^2 - s^2}\right) 2s \frac{m(\mu_k)}{\mu_k + s^2}.$$

We multiply both sides of (7.36) by

$$\prod_{j=2}^{N} (s_j^2 - s^2).$$

Then, the residues of $L_S(s)$ at the points $\pm i\sqrt{\mu_k}$ are $m(\mu_k)$, for $k \neq 0$ and $2m(0)$, for $k = 0$. By (3.24), $L_S(s)$ decreases exponentially as $\mathrm{Re}(s) \to \infty$. Therefore, the integral

$$\int_s^{\infty} L_S(w)\mathrm{d}w$$

over a path connecting $s$ and infinity is well defined and

$$\log S(s;\sigma,\chi) = -\int_s^{\infty} L_S(w)\mathrm{d}w. \tag{7.38}$$

The integral above depends on the choice of the path, because $L_S(s)$ has singularities. Since the residues of the singularities are integers, we can use the same argument as in the proof of Theorem 7.8. If we exponentiate the right-hand side of (7.38), then this exponential is independent of the choice of the path. The meromorphic continuation of the symmetrized zeta function $S(s;\sigma,\chi)$ to the whole complex plane follows. □

**Theorem 7.11** *The Selberg zeta function $Z(s;\sigma,\chi)$ admits a meromorphic continuation to the whole complex plane $\mathbb{C}$. The set of the singularities equals to $\{s_k^{\pm} = \pm i\lambda_k : \lambda_k \in$ $\mathrm{spec}(D_\chi^\sharp(\sigma)), k \in \mathbb{N}\}$. The orders of the singularities are equal to $\frac{1}{2}(\pm m_s(\lambda_k) + m(\lambda_k^2)$. For $\lambda_0 = 0$, the order of the singularity is equal to $m(0)$.*

*Proof* We observe at first that

$$Z(s;\sigma,\chi) = \sqrt{S(s;\sigma,\chi)Z^s(s;\sigma,\chi)}.$$

Recall that by Eq. (6.8) we have $A_\chi^\sharp(\sigma) = D_\chi^\sharp(\sigma)^2$. Hence, we can identify the eigenvalues $\mu_k$ of $A_\chi^\sharp(\sigma)$ with $\lambda_k^2$, where $\lambda_k \in \mathrm{spec}(D_\chi^\sharp(\sigma))$. By Theorem 7.8 and Theorem 7.10, the product $S(s;\sigma,\chi)Z^s(s,\sigma,\chi)$ has its singularities at $s_k^{\pm} = \pm i\lambda_k$, of order $\pm m_s(\lambda_k) + m(\lambda_k^2)$. We need to prove that the order of the singularities of $Z(s;\sigma,\chi)$ is an even integer. This follows from the definition of the algebraic multiplicities $m_s(\lambda_k)$, $m(\lambda_k^2)$ and the construction of the locally homogeneous vector bundles $E(\sigma)$ and $E_{\tau_s(\sigma)}$ associated with the representations $\tau_\sigma$ and $\tau_s(\sigma)$ of $K$. By Proposition 5.4 together with Eqs. (5.18) and (5.19), $E(\sigma)=E_{\tau_s(\sigma)}$ up to a $\mathbb{Z}_2$-grading. Hence, $m_s(\lambda_k)\equiv m(\lambda_k^2) \mod 2$. The assertion follows. □

### 7.5 Meromorphic continuation of the Ruelle zeta function

In this section we will prove the meromorphic continuation of the Ruelle zeta function to the whole complex plane. In view of the previous results, from Sects. 7.3 and 7.4, we will use the meromorphic continuation of the Selberg zeta function. Furthermore, we will use Theorem 7.12 below, which provides a representation of the Ruelle zeta function as a product of Selberg zeta functions with shifted arguments.

We consider the following identification $\mathfrak{a}_{\mathbb{C}}^*\cong\mathbb{C}\ni\lambda$. Let $\alpha>0$ be the unique positive root of the system $(\mathfrak{g},\mathfrak{a})$. Let $\lambda\colon A\to\mathbb{C}^\times$ be the character, defined by $\lambda(a)=e^{\alpha(\log a)}$.

Let $\mathfrak{n}_{\mathbb{C}}$ be the complexification of the Lie algebra $\mathfrak{n}$. Let $\nu_p:=\Lambda^p\operatorname{Ad}_{\mathfrak{n}_{\mathbb{C}}}(MA)$ be the representation of $MA$ in $\Lambda^p\mathfrak{n}_{\mathbb{C}}$ given by the $p$th exterior power of the adjoint representation:

$$\nu_p:=\Lambda^p\operatorname{Ad}_{\mathfrak{n}_{\mathbb{C}}}\colon MA\to\operatorname{GL}(\Lambda^p\mathfrak{n}_{\mathbb{C}}),\quad p=0,1,\ldots,d-1.$$

For $p=0,1,\ldots,d-1$, let $J_p\subset\{(\psi_p,\lambda)\colon\psi_p\in\widehat{M},\lambda\in\mathbb{C}\}$ be the subset consisting of all pairs of unitary irreducible representations of $M$ and one-dimensional representations of $A$ such that, as $MA$-modules, the representations $\nu_p$ decompose as

$$\Lambda^p\mathfrak{n}_{\mathbb{C}}=\bigoplus_{(\psi_p,\lambda)\in J_p}V_{\psi_p}\otimes\mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda\cong\mathbb{C}$ denotes the representation space of $\lambda$.

For $\sigma\in\widehat{M}$ we define

$$Z_p(s;\sigma,\chi):=\prod_{(\psi_p,\lambda)\in J_p}Z(s+|\rho|-\lambda;\psi_p\otimes\sigma,\chi). \tag{7.39}$$

**Theorem 7.12** *Let $\sigma\in\widehat{M}$. Then, the Ruelle zeta function has the representation*

$$R(s;\sigma,\chi)=\prod_{p=0}^{d-1}Z_p(s;\sigma,\chi)^{(-1)^p}. \tag{7.40}$$

*Proof* By (3.14), we have

$$\log Z(s+|\rho|-\lambda;\psi_p\otimes\sigma,\chi)$$
$$=-\sum_{[\gamma]\neq e}\frac{1}{n_\Gamma(\gamma)}\operatorname{tr}(\chi(\gamma)\otimes\sigma(m_\gamma))\frac{e^{(-s-2|\rho|+\lambda)l(\gamma)}\operatorname{tr}(\psi_p(m))}{\det(\operatorname{Id}-\operatorname{Ad}(m_\gamma a_\gamma)_{\overline{\mathfrak{n}}})}. \tag{7.41}$$

We use now the fact that

$$\det(\operatorname{Id}-\operatorname{Ad}(m_\gamma a_\gamma)_{\overline{\mathfrak{n}}})=(-1)^{d-1}a_\gamma^{-2|\rho|}\det(\operatorname{Id}-\operatorname{Ad}(m_\gamma a_\gamma)_{\mathfrak{n}}) \tag{7.42}$$

(see [2, p. 93]). Hence, (7.41) becomes from (7.42),

$$\log Z(s+|\rho|-\lambda;\psi_p\otimes\sigma,\chi)$$
$$=(-1)^d\sum_{[\gamma]\neq e}\frac{1}{n_\Gamma(\gamma)}\operatorname{tr}(\chi(\gamma)\otimes\sigma(m_\gamma))\frac{e^{-sl(\gamma)}e^{\lambda l(\gamma)}\operatorname{tr}(\psi_p(m))}{\det(\operatorname{Id}-\operatorname{Ad}(m_\gamma a_\gamma)_{\mathfrak{n}})}. \tag{7.43}$$

We have

$$
\begin{aligned}
\log \prod_{p=0}^{d-1} Z_p(s;\sigma,\chi)^{(-1)^p} &= \sum_{p=0}^{d-1} \log Z_p(s;\sigma,\chi)^{(-1)^p}\\
&= \sum_{p=0}^{d-1} (-1)^p \log Z_p(s;\sigma,\chi)\\
&= \sum_{p=0}^{d-1} (-1)^p \log \prod_{(\psi_p,\lambda)\in J_p} Z(s+|\rho|-\lambda;\psi_p\otimes\sigma,\chi)\\
&= \sum_{p=0}^{d-1} (-1)^p \sum_{(\psi_p,\lambda)\in J_p} \log\big(Z(s+|\rho|-\lambda;\psi_p\otimes\sigma,\chi)\big)\\
&= \sum_{p=0}^{d-1} (-1)^p \Bigg( (-1)^d \sum_{[\gamma]\neq e} \frac{1}{n_\Gamma(\gamma)} \operatorname{tr}(\chi(\gamma)\otimes\sigma(m_\gamma))e^{-sl(\gamma)}\\
&\qquad \frac{\sum_{(\psi_p,\lambda)\in J_p} e^{\lambda l(\gamma)} \operatorname{tr}(\psi_p(m))}{\det(\mathrm{Id}-\mathrm{Ad}(m_\gamma a_\gamma)_{\mathfrak{n}})}\Bigg), \qquad (7.44)
\end{aligned}
$$

where in the third line of Eq. (7.44) we used Eq. (7.39) and in the last line we used Eq. (7.43). We recall now the that for any endomorphism of a finite dimensional vector space $W$, we have

$$\det(\mathrm{Id}_W - W) = \sum_{p=0}^{\infty} (-1)^p \operatorname{tr}(\Lambda^p W).$$

If we apply this identity to $\mathrm{Ad}(m_\gamma a_\gamma)_{\mathfrak{n}}$ we get

$$\sum_{p=0}^{d-1} (-1)^p \frac{\sum_{(\psi_p,\lambda)\in J_p} e^{\lambda l(\gamma)} \operatorname{tr}(\psi_p(m))}{\det(\mathrm{Id}-\mathrm{Ad}(m_\gamma a_\gamma)_{\mathfrak{n}})} = 1.$$

Hence, by (7.44) and (3.17), we have

$$
\begin{aligned}
\log \prod_{p=0}^{d-1} Z_p(s;\sigma,\chi)^{(-1)^p} &= (-1)^d \sum_{[\gamma]\neq e} \frac{1}{n_\Gamma(\gamma)} \operatorname{tr}(\chi(\gamma)\otimes\sigma(m_\gamma))e^{-sl(\gamma)}\\
&= \log R(s;\sigma,\chi).
\end{aligned}
$$

□

**Theorem 7.13** *For every $\sigma \in \widehat{M}$, the Ruelle zeta function $R(s;\sigma,\chi)$ admits a meromorphic continuation to the whole complex plane $\mathbb{C}$.*

*Proof*
- If $\sigma \in \widehat{M}$ is Weyl invariant, the assertion follows from Theorem 7.9 together with Theorem 7.12.
- If $\sigma \in \widehat{M}$ is non-Weyl invariant, the assertion follows from Theorem 7.11 together with Theorem 7.12. □

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## 8 Appendix

In Sects. 5 and 6, we define the twisted Bochner–Laplace and Dirac operator, respectively. These operators are associated with a non-unitary representation of the subgroup $\Gamma$ and are no longer self-adjoint. However, the twisted operators $\Delta^{\sharp}_{\tau,\chi}$ and $D^{\sharp}_{\chi}(\sigma)^2$ have the same principal symbol as the operators $\Delta_\tau \otimes \mathrm{Id}_{V_\chi}$ and $D(\sigma)^2 \otimes \mathrm{Id}_{V_\chi}$, respectively, i.e.,

$$\sigma_{\Delta^{\sharp}_{\tau,\chi}}(x,\xi) = \|\xi\|^2 \,\mathrm{Id}_{(V_\tau \otimes V_\chi)_x},$$
$$\sigma_{D^{\sharp}_{\chi}(\sigma)^2}(x,\xi) = \|\xi\|^2 \,\mathrm{Id}_{(V_{\tau_s(\sigma)} \otimes V_\chi)_x},$$

where $x \in X$, $\xi \in \mathrm{T}^*_x X$, $\xi \neq 0$, We include here the the spectral theory of non-self-adjoint operators, which is needed to develop the trace formulas and further to provide the proofs of the meromorphic continuations of the twisted dynamical zeta functions.

**Setting 8.1** *Let $E \to X$ be a complex vector bundle over a smooth compact Riemannian manifold $X$ of dimension $d$. Let $D : C^\infty(X,E) \to C^\infty(X,E)$ be an elliptic differential operator of order $m \geq 1$. Let $\sigma_D$ be its principal symbol.*

**Definition 8.2** A spectral cut is a ray

$$R_\theta := \left\{\rho e^{i\theta} : \rho \in [0,\infty]\right\},$$

where $\theta \in [0, 2\pi)$.

**Definition 8.3** The angle $\theta$ is a principal angle for an elliptic operator $D$ if

$$\mathrm{spec}(\sigma_D(x,\xi)) \cap R_\theta = \emptyset, \quad \forall x \in X, \forall \xi \in T^*_x X, \xi \neq 0.$$

**Definition 8.4** We define the solid angle $L_I$ associated with a closed interval $I$ of $\mathbb{R}$ by

$$L_I := \{\rho e^{i\theta} : \rho \in (0,\infty), \theta \in I\}.$$

**Definition 8.5** The angle $\theta$ is an Agmon angle for an elliptic operator $D$, if it is a principal angle for $D$ and there exists $\varepsilon > 0$ such that

$$\mathrm{spec}(D) \cap L_{[\theta-\varepsilon,\theta+\varepsilon]} = \emptyset.$$

**Lemma 8.6** *Let $\varepsilon$ be an angle such that the principal symbol $\sigma_D(x,\xi)$ of $D$, for $\xi \in T^*_x X$, $\xi \neq 0$, does not take values in $L_{[-\varepsilon,\varepsilon]}$. Then, the spectrum* spec$(D)$ *of the operator $D$ is discrete and for every $\varepsilon \in (0, \frac{\pi}{2})$ there exists $R > 0$ such that* spec$(D)$ *is contained in the set $B(0,R) \cup L_{[-\varepsilon,\varepsilon]} \subset \mathbb{C}$.*

*Proof* The discreteness of the spectrum follows from [26, Theorem 8.4]. For the second statement see [26, Theorem 9.3].

Let $\lambda_k$ be an eigenvalue of $D$ and $V_{\lambda_k}$ be the corresponding eigenspace. This is a finite dimensional subspace of $C^\infty(X,E)$ invariant under $D$. We have that for every $k \in \mathbb{N}$, there exist $N_k \in \mathbb{N}$ such that

$$(D - \lambda_k \,\mathrm{Id})^{N_k} V_{\lambda_k} = 0;$$
$$\lim_{k\to\infty} |\lambda_k| = \infty.$$

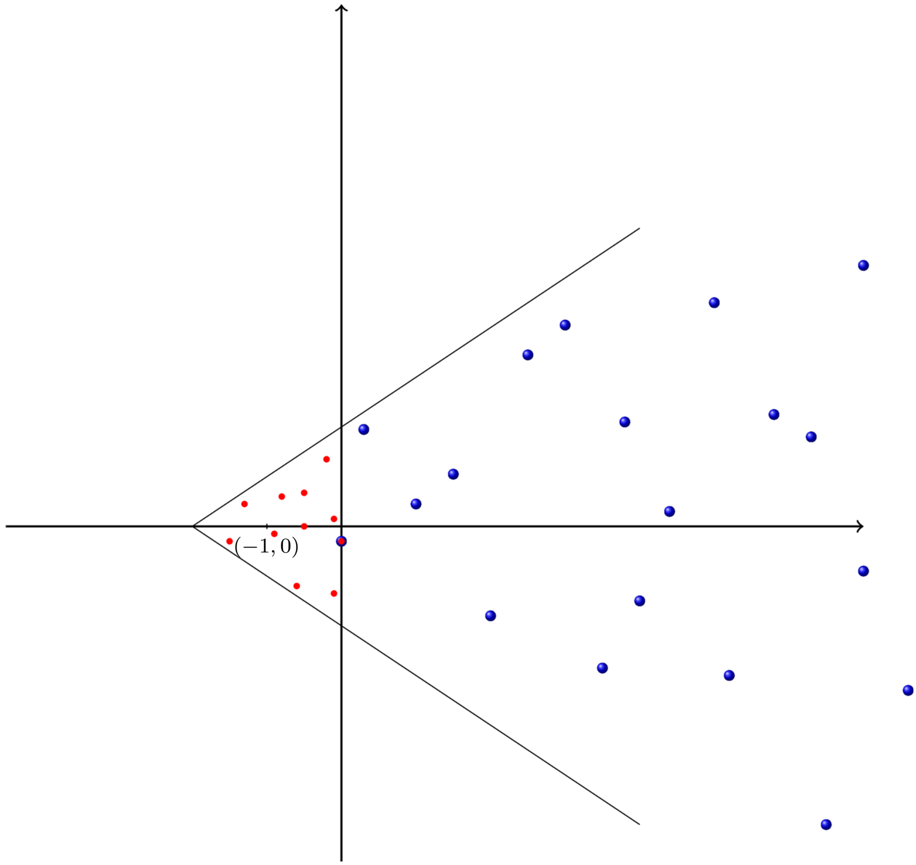


**Fig. 1** The discrete spectrum of the operator $D^{\sharp}_{\chi}(\sigma)^2$

By [16] the space $L^2(X, E)$ can be decomposed as

$$L^2(X, E) = \overline{\bigoplus_{k\geq 1} V_{\lambda_k}}.$$

This is the generalization of the eigenspace decomposition of a self-adjoint operator.

**Definition 8.7** We call algebraic multiplicity $m(\lambda_k)$ of the eigenvalue $\lambda_k$ the dimension of the corresponding eigenspace $V_{\lambda_k}$.

By Lemma 8.6, there exists $\varepsilon > 0$ such that

$$\mathrm{spec}(D^{\sharp}_{\chi}(\sigma)^2) \cap L_{[\theta-\varepsilon,\theta+\varepsilon]} = \emptyset.$$

Since $D^{\sharp}_{\chi}(\sigma)^2$ has discrete spectrum (Fig. 1), there exists also an $r_0 > 0$ such that

$$\mathrm{spec}(D^{\sharp}_{\chi}(\sigma)^2) \cap \{z \in \mathbb{C} : |z+1| \leq 2r_0\} = \emptyset.$$

We define a contour $\Gamma_{\theta,r_0}$ as follows.

$$\Gamma_{\theta,r_0} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3,$$

where $\Gamma_1 = \{-1 + re^{i\theta} : \infty > r \geq r_0\}$, $\Gamma_2 = \{-1 + r_0 e^{ia} : \theta \leq a \leq \theta + 2\pi\}$, $\Gamma_3 = \{-1 + re^{i(\theta+2\pi)} : r_0 \leq r < \infty\}$. On $\Gamma_1$, $r$ runs from $\infty$ to $r_0$, $\Gamma_2$ is oriented counterclockwise, and on $\Gamma_3$, $r$ runs from $r_0$ to $\infty$. We put

$$e^{-tD^{\sharp}_{\chi}(\sigma)^2} = \frac{i}{2\pi} \int_{\Gamma_{\theta,r_0}} e^{-t\lambda} \big(D^{\sharp}_{\chi}(\sigma)^2 - \lambda \,\mathrm{Id}\big)^{-1} \mathrm{d}\lambda. \tag{8.1}$$

We have $|e^{-t\lambda}| \leq e^{-t\mathrm{Re}(\lambda)}$. Furthermore, by [26, Corollary 9.2], there exists a positive constant $c > 0$ such that $\|(D^{\sharp}_{\chi}(\sigma)^2 - \lambda \,\mathrm{Id})^{-1}\| \leq c|\lambda|^{-1}$, for $\lambda \in \Gamma_{\theta,r_0}$. Hence, the integral in (8.1) are well defined.

Given an Agmon angle $\theta$ for the operator $A^{\sharp}_{\chi}(\sigma)$ (see equations (5.24)–(5.26), p. 29) and $r_0 > 0$, we consider a contour $\Gamma_{\theta,r_0}$ in the same way as for the operator $D^{\sharp}_{\chi}(\sigma)^2$. Then, we put

$$e^{-tA^{\sharp}_{\chi}(\sigma)} = \frac{i}{2\pi} \int_{\Gamma_{\theta,r_0}} e^{-t\lambda} (A^{\sharp}_{\chi}(\sigma) - \lambda \,\mathrm{Id})^{-1} \mathrm{d}\lambda. \tag{8.2}$$

By [26, Corollary 9.2] and the fact that $|e^{-t\lambda}| \leq e^{-t\mathrm{Re}(\lambda)}$, the integral in (8.2) is well defined.

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