

Twisted Ruelle zeta function at zero for compact hyperbolic surfaces

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May 28, 2021

Abstract

Let X be a compact, hyperbolic surface of genus $g \geq 2$. In this paper, we prove that the twisted Selberg and Ruelle zeta functions, associated with an arbitrary, finite-dimensional, complex representation χ of $\pi_1(X)$ admit a meromorphic continuation to \mathbb{C} . Moreover, we study the behaviour of the twisted Ruelle zeta function at $s = 0$ and prove that at this point, it has a zero of order $\dim(\chi)(2g - 2)$.

Keywords — twisted Selberg zeta function, twisted Ruelle zeta function, non-unitary representations, Selberg trace formula.

Mathematics Subject Classification (2020) — Primary: 11M36; Secondary: 11F72, 37C30.

1 Introduction

The Selberg zeta function has been introduced by Selberg in [31]. For a hyperbolic surface, it is defined by an Euler-type product over the prime closed geodesics, i.e., the closed geodesics which are not multiples of shorter geodesics, and in terms of the lengths of closed geodesics on the surface, also known as length spectrum. Let Γ be a discrete, torsion-free, cocompact subgroup of $\mathrm{PSL}_2(\mathbb{R})$ and let $X = \Gamma \backslash \mathbb{H}^2$ be the associated compact hyperbolic surface. The goal of this paper is to prove that the Selberg and Ruelle zeta functions for X , associated with a non-unitary representation of $\Gamma \cong \pi_1(X)$, admit a meromorphic continuation to \mathbb{C} . In addition, we study the behaviour of the twisted Ruelle zeta function near the origin and prove that it is related to a topological invariant, the Euler characteristic of the surface.

We define the *twisted* dynamical zeta functions, associated with a finite-dimensional, complex representation $\chi: \Gamma \rightarrow \mathrm{GL}(V_\chi)$ of Γ , which is not necessarily unitary. For $s \in \mathbb{C}$, the twisted Selberg zeta function, associated with χ , is given by the infinite product

$$Z(s; \chi) := \prod_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \prod_{k=0}^{\infty} \det(\mathrm{Id} - \chi(\gamma) e^{-(s+k)l(\gamma)}),$$

and the twisted Ruelle zeta function, associated with χ , is defined by the infinite product

$$R(s; \chi) := \prod_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \det(\text{Id} - \chi(\gamma)e^{-sl(\gamma)}).$$

Here, the products run over the primitive conjugacy classes of Γ (see Subsection 2.1), which correspond to the prime closed geodesics on the surface with length $l(\gamma)$.

The *twisted* Selberg, resp. Ruelle zeta function, converges absolutely and uniformly on compact subsets of some half plane in \mathbb{C} (Proposition 4.1.1, resp. 4.1.3). If we consider χ to be the trivial representation, we obtain the usual, *non-twisted* definitions of the Selberg and Ruelle zeta functions (see [29, 31]).

Fried, in [15], studied the case of a compact hyperbolic orbisurface $X = \Gamma \backslash \mathbb{H}^2$. He proved that for a unitary representation ρ of $\pi_1(S(X))$, the fundamental group of the unit sphere bundle $S(X)$ of X , which factors through Γ , the Ruelle zeta function $R(s; \rho)$ associated with ρ vanishes at zero to the order given by the Euler characteristic times the dimension of ρ ([15, Corollary 2]). Dyatlov and Zworski in [9] generalized this result to the case of a negatively curved oriented surface, using techniques from semiclassical analysis ([9, Theorem p. 212]). Concerning 2-dimensional manifolds, we mention here also the recent paper of Riviere and Dang [5], where an object closely related to the dynamical zeta functions, the generalized Poincaré series is defined, and the work of Fedosova and Pohl in [12] for the Selberg zeta function for geometrically finite Fuchsian groups Γ , twisted by finite-dimensional representations with non-expanding cusp monodromy. The special value at zero of the Ruelle zeta function has been studied in relation with the so called Fried conjecture, which asks whether or not the Ruelle zeta function is analytic at zero and related to spectral and topological invariants. Fried's conjecture has been solved in different geometric and algebraic settings, see [3, 4, 14, 23, 24, 27, 32, 34, 33, 39]. The algebraic geometric analogue to Fried's conjecture is the Lichtenbaum conjecture, which concerns special values of the Hasse–Weil zeta function for regular schemes, as it is described in [7].

The dynamical zeta functions associated with a non-unitary representation of the fundamental group of a compact, odd dimensional, hyperbolic manifold has been studied in [37] and [38]. In these papers, the second author proved the meromorphic continuation and the functional equations for the dynamical zeta functions, using as a main tool the Selberg trace formula for non-unitary twists as it is introduced in [26]. Müller, in [26], introduced a Selberg trace formula for suitable test functions, the even Paley-Wiener functions, and non-unitary representations of the lattice Γ of a locally symmetric space $X = \Gamma \backslash G/K$ of real rank 1 ([26, Theorem 1.1]). In the present paper we apply this trace formula, but for the heat operator of a twisted Laplacian Δ_χ^\sharp as in [37]. The twisted Laplace operator Δ_χ^\sharp acts on the space of smooth sections of the non-unitary flat vector bundle over X associated with χ (see Subsection 3.1) and is no longer self-adjoint. Nevertheless, it is in elliptic operator and hence it has nice spectral properties (for further details see Subsection 3.1). We summarize here the main results of the paper.

Theorem A (see Theorem 4.2.6). *Let $X = \Gamma \backslash \mathbb{H}^2$ be a compact hyperbolic surface and let $\chi: \Gamma \rightarrow \text{GL}(V_\chi)$ be a finite-dimensional, complex representation*

of Γ . Then, the twisted Selberg zeta function $Z(s; \chi)$ admits a meromorphic continuation to \mathbb{C} . Moreover, the zeros/singularities of $Z(s; \chi)$ are contained in the set $\{\frac{1}{2} \pm i\mu_j : j \in \mathbb{Z}^+\} \cup \mathbb{Z}^-$, where $\mathbb{Z}^\pm = \{0, \pm 1, \pm 2, \dots\}$ and $\text{spec}(\Delta_\chi^\sharp) = \{\frac{1}{4} + \mu_j^2\}_{j \in \mathbb{Z}^+} \subseteq \mathbb{C}$ are the eigenvalues of the twisted Laplacian Δ_χ^\sharp .

Using the fundamental relation between the Selberg and the Ruelle zeta function, we obtain:

Corollary B (see Corollary 4.2.8). *The twisted Ruelle zeta function $R(s; \chi)$ admits a meromorphic continuation to \mathbb{C} .*

Applying additionally the functional equation for the twisted Selberg zeta function yields:

Corollary C (see Corollary 4.3.1). *The twisted Ruelle zeta function $R(s; \chi)$ near $s = 0$ is given by*

$$(1.1) \quad R(s; \chi) = \pm(2\pi s)^{\dim(V_\chi)(2g-2)} + \text{higher order terms.}$$

As mentioned above, the main ingredient in the proof of the meromorphic continuation of the twisted Selberg zeta function is the Selberg trace formula for suitable integral operators, namely the heat operators, induced by the twisted Laplace operator Δ_χ^\sharp (see Subsection 3.2). In fact, a resolvent trace formula is utilized to obtain the meromorphic continuation of the twisted Selberg zeta function, see Lemma 4.1.2 and Proposition 4.2.4. Hence, by Lemma 4.1.5, the meromorphic continuation of the twisted Ruelle zeta function follows. Further, we obtain the functional equation for the twisted Selberg zeta function (Theorem 4.2.9). This is the key point to prove Corollary C. This result can be viewed as the extension of the result of Fried ([15, Corollary 2]) to the case of a non-unitary representation of Γ . In [15], Fried considered a Fuchsian group Γ , allowing also elliptic elements. In such a case, $X = \Gamma \backslash \mathbb{H}^2$ is a compact orbisurface and the unit sphere bundle $S(X) = \Gamma \backslash \text{PSL}_2(\mathbb{R})$ is a Seifert fiber space over X , which can be viewed as a 3-manifold. Fried considered the twisted Ruelle zeta function $R_\rho(s)$, associated with ρ , where ρ is a unitary representation of $\pi_1(S(X))$. It holds $\pi_1(X) \simeq \pi_1(S(X))/\mathbb{Z}$. By [15, Theorem 3], for an acyclic, unitary representation ρ of $\pi_1(S(X))$,

$$(1.2) \quad |R_\rho(0)|^{-1} = \tau_\rho,$$

where τ_ρ is the Reidemeister torsion ([6, 13, 28]) of $S(X)$. A natural question is if one can extend these results to the case of a non-unitary representation of $\pi_1(S(X))$, which is classical in the sense of [15, p. 149]. We hope to deal with this problem in a future work.

Organization of the paper. In Section 2, we review well-known theory of the geometry of hyperbolic surfaces, the Laplace–Beltrami operator, and the principal series representation of $\text{PSL}_2(\mathbb{R})$. In Section 3, we introduce the twisted Bochner–Laplace operator acting on the space of sections of a vector bundle over X and recall its spectral properties. In the same section, we prove the trace formula for the heat operator induced by the twisted Laplacian. Section 4 is the core of this article. In this section, we define the Selberg and Ruelle zeta function associated with a non-unitary representation of the fundamental group

of X and prove that they admit a meromorphic continuation to \mathbb{C} . Moreover, we prove functional equations for the twisted Selberg function, which further lead to the result for the behaviour of the twisted Ruelle zeta function at the origin.

Acknowledgements. The authors would like to thank the the Department of Mathematics at Aarhus University, where this work has been conducted, for its hospitality. In addition, the authors would like to thank Werner Müller and Léo Bénard for helpful discussions and comments about the results of Fried in [15]. Both authors were supported by a research grant from the Villum Foundation (Grant No. 00025373).

2 Preliminaries

In this section we set up some notation around the geometry and analysis on hyperbolic surfaces and representation theory of $\mathrm{PSL}(2, \mathbb{R})$.

2.1 Hyperbolic surfaces

We consider the upper half plane

$$\mathbb{H}^2 := \{z = x + iy : y > 0\}.$$

Then, $G = \mathrm{PSL}(2, \mathbb{R})$ acts on \mathbb{H}^2 by fractional linear transformations. This action is transitive and the maximal compact subgroup $K = \mathrm{PSO}(2)$ is the stabilizer of $i \in \mathbb{H}^2$, hence $\mathbb{H}^2 \cong G/K = \mathrm{PSL}(2, \mathbb{R})/\mathrm{PSO}(2)$. The G -invariant metric induced by the restriction of the Killing form on the Lie algebra \mathfrak{g} to the Cartan complement \mathfrak{p} of the Lie algebra \mathfrak{k} of K is the Poincaré metric

$$(2.1) \quad ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

Let $G = NAK$ be the Iwasawa decomposition of G , where

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}, \quad \text{and} \quad A = \left\{ \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} : y > 0 \right\},$$

and write \mathfrak{n} and \mathfrak{a} for the Lie algebras of N and A . The elements of A are the hyperbolic elements and they act on \mathbb{H}^2 by dilations by $p > 0$, i.e., $z \mapsto pz$. The unit sphere bundle $S(\mathbb{H}^2)$ of \mathbb{H}^2 can be identified with the group $G = \mathrm{PSL}(2, \mathbb{R})$ via the quotient map $G \rightarrow G/K$. The geodesic flow on $S(\mathbb{H}^2)$ is given by the right action of the one parameter group A . For more details about the classical theory of \mathbb{H}^2 see e.g. [10, 21, 25].

Now let Γ be a Fuchsian group of the first kind, that is a finitely generated discrete subgroup of G , such that $\Gamma \backslash G$ is of finite volume. We assume that Γ is torsion-free, i.e., there are no elements of finite order. Moreover, we assume that Γ is cocompact. Hence, Γ contains only hyperbolic elements. Then, $X = \Gamma \backslash \mathbb{H}^2 = \Gamma \backslash G/K$ is a compact, hyperbolic surface of genus $g \geq 2$, and conversely every compact, hyperbolic surface arises in this way.

The Γ -conjugacy classes $[\gamma]$ of elements $\gamma \in \Gamma$ correspond to the closed geodesics in X in the sense that every $\gamma \in \Gamma = \pi_1(X)$ is represented by a unique

closed geodesic. A closed geodesic is called prime if it cannot be expressed as the multiple of a shorter closed geodesic. For the conjugacy class $[\gamma]$ this is equivalent to γ being primitive, i.e., γ cannot be written as $\gamma = \gamma_0^n$ with $\gamma_0 \in \Gamma$, $n \geq 2$. Every non-trivial $\gamma \in \Gamma$ can be written as $\gamma = \gamma_0^n$ with $\gamma_0 \in \Gamma$ primitive and $n \geq 1$, and $l(\gamma) = n \cdot l(\gamma_0)$. We use the notation $n_\Gamma(\gamma) = n$.

Every $\gamma \in \Gamma$, $\gamma \neq e$, is hyperbolic and hence conjugate to an element $a_\gamma \in A$. Note that $\text{Ad}(a_\gamma)|_{\mathfrak{n}} = e^{l(\gamma)} \text{Id}_{\mathfrak{n}}$.

2.2 The Laplace–Beltrami operator

Let

$$(2.2) \quad \tilde{\Delta} = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right),$$

denote the Laplace–Beltrami operator on \mathbb{H}^2 , where we write $z = x + iy \in \mathbb{H}^2$. If we define the left translation of a smooth function f on \mathbb{H}^2 by $g \in G$ as

$$L_g f(z) := f(g^{-1}z),$$

Then, for every $g \in G$

$$L_g \tilde{\Delta} = \tilde{\Delta} L_g,$$

so $\tilde{\Delta}$ descends to a differential operator Δ on $\Gamma \backslash \mathbb{H}^2$.

2.3 Unitary principal series representations

Let $P = AN$ denote the standard parabolic subgroup of G . We identify $\mathfrak{a}_{\mathbb{C}}^* \simeq \mathbb{C}$ by $\lambda \mapsto \lambda(\text{diag}(\frac{1}{2}, -\frac{1}{2}))$. Then $\rho = \frac{1}{2} \text{tr ad}|_{\mathfrak{n}}$ corresponds to $\frac{1}{2}$. For $\lambda \in \mathbb{R}$ we form the representation $e^{i\lambda} \otimes 1$ of $P = AN$ and consider the induced representation

$$\pi_\lambda = \text{Ind}_P^G(e^{i\lambda} \otimes 1)$$

of G on the Hilbert space \mathcal{H}_λ of L^2 -sections of the homogeneous vector bundle $G \times_P (e^{i\lambda + \rho} \otimes 1) \rightarrow G/P$. These representations are unitary and irreducible for all $\lambda \in \mathbb{R}$ and are called the *unitary principal series*. We write Θ_λ for the *distribution character* of π_λ defined by

$$\Theta_\lambda(\varphi) = \text{tr} \int_G \varphi(g) \pi_\lambda(g) dg \quad (\varphi \in C_c^\infty(G)),$$

where dg denotes a suitably normalized Haar measure on G .

We denote by \mathcal{H}_λ^K the subspace of \mathcal{H}_λ of $v \in \mathcal{H}_\lambda$ such that $\pi(k)v = v$, for all $k \in K$.

3 Harmonic analysis on hyperbolic surfaces

Following Müller [26], we describe the construction of the twisted Bochner–Laplace operator on vector bundles over $X = \Gamma \backslash \mathbb{H}^2$ induced from non-unitary representations χ of Γ and use it to obtain a trace formula for X .

3.1 The twisted Bochner–Laplace operator

In this section, we define the twisted Bochner–Laplace operator as introduced in [26, Section 4]. This operator acts on sections of twisted vector bundles. It is an elliptic operator, but longer self-adjoint. Nevertheless, it has a self-adjoint principle symbol and hence it has qualitatively similar spectral properties.

Let χ be a finite-dimensional, complex representation

$$\chi: \Gamma \rightarrow \mathrm{GL}(V_\chi)$$

of Γ . Let $E_\chi = V_\chi \times_\Gamma X \rightarrow X$ be the associated flat vector bundle over X , equipped with a flat connection ∇^{E_χ} . We recall the construction of the twisted Bochner–Laplace operator Δ_χ^\sharp , acting on smooth sections of E_χ .

Remark 3.1.1. The definition of the twisted Bochner–Laplace operator in the present paper is the same as in [26, Section 4] for τ being the trivial representation.

The second covariant derivative $(\nabla^{E_\chi})^2$ is defined by

$$(\nabla^{E_\chi})^2_{V,W} := \nabla_V^{E_\chi} \nabla_W^{E_\chi} - \nabla_{\nabla_V^{LC} W}^{E_\chi},$$

where $V, W \in C^\infty(X, TX)$, TX is the tangent bundle of X , and ∇^{LC} denotes the Levi–Civita connection on TX . The twisted Bochner–Laplace operator Δ_χ^\sharp is defined to be the corresponding connection Laplacian on E_χ , i.e. the negative of the trace of the second covariant derivative:

$$(3.1) \quad \Delta_\chi^\sharp := -\mathrm{tr}((\nabla^{E_\chi})^2).$$

Locally, this operator is described as follows. We consider an open subset U of X such that $E_\chi|_U$ is trivial, i.e., $E_\chi|_U \cong U \times \mathbb{C}^m$, where $m = \mathrm{rank}(E_\chi) = \dim V_\chi$. Let e_1, \dots, e_m be any basis of flat sections of $E_\chi|_U$. Then, each $\phi \in C^\infty(U, E_\chi|_U)$ can be written as $\phi = \sum_{i=1}^m \phi_i \otimes e_i$, where $\phi_i \in C^\infty(U)$, $i = 1, \dots, m$. Then,

$$(3.2) \quad \nabla_Y^{E_\chi} \phi = \sum_{i=1}^m \nabla_Y \phi_i \otimes e_i, \quad (Y \in C^\infty(X, TX)).$$

The local expression above is independent of the choice of the basis of flat sections of $E_\chi|_U$, since the transition maps comparing flat sections are constant. By (3.1) and (3.2), the twisted Bochner–Laplace operator acting on $C^\infty(U, E_\chi|_U)$ is given by

$$(3.3) \quad \Delta_\chi^\sharp \phi = \sum_{i=1}^m (\Delta \phi_i) \otimes e_i,$$

where Δ denotes the Laplace–Beltrami operator on X .

Let now \tilde{E}_χ be the pullback to $\tilde{X} = \mathbb{H}^2$ of E_χ . Then,

$$\tilde{E}_\chi \cong \tilde{X} \times V_\chi,$$

and

$$(3.4) \quad C^\infty(\tilde{X}, \tilde{E}_\chi) \cong C^\infty(\tilde{X}) \otimes V_\chi.$$

With respect to the isomorphism (3.4), it follows from (3.3) that the lift $\tilde{\Delta}_\chi^\sharp$ of Δ_χ^\sharp to $\tilde{X} = \mathbb{H}^2$ takes the form

$$(3.5) \quad \tilde{\Delta}_\chi^\sharp = \tilde{\Delta} \otimes \text{Id}_{V_\chi},$$

where $\tilde{\Delta}$ is the Laplace–Beltrami operator on \mathbb{H}^2 .

If we choose a Hermitian metric on E_χ , then Δ_χ^\sharp acts in $L^2(X, E_\chi)$ with domain $C^\infty(X, E_\chi)$. However, it is not a formally self-adjoint operator in general. By (3.3), Δ_χ^\sharp has principal symbol

$$\sigma_{\Delta_\chi^\sharp}(x, \xi) = \|\xi\|_x^2 \text{Id}_{(E_\chi)_x} \quad (x \in X, \xi \in T_x^*X).$$

Hence, Δ_χ^\sharp is an elliptic, second order differential operator with the following spectral properties: its spectrum is discrete and contained in a translate of a positive cone $C \subset \mathbb{C}$ such that $\mathbb{R}^+ \subset C$. This fact follows from classical spectral theory of elliptic operators, under the assumption of the compactness of the manifold. We refer the reader to [35], and also [26, Lemma 2.1]. Moreover, the direct sum of all generalized eigenspaces is dense in $L^2(X, E_\chi)$.

3.2 The trace formula

For the derivation of the trace formula, we follow the classical work of Walsch [40], and its extension to non-unitary representations of Γ by Müller [26]. Restricting to the case of dimension 2 and the trivial representation τ of K , we are led to a trace formula, which will be the basic tool to prove the meromorphic continuation of the twisted dynamical zeta functions in the next section.

We denote by $\text{spec}(\Delta_\chi^\sharp)$ the (discrete) spectrum of Δ_χ^\sharp . For $\mu \in \text{spec}(\Delta_\chi^\sharp)$, we write $L^2(X, E_\chi)_\mu$ for the corresponding generalized eigenspace. We define the algebraic multiplicity $m(\mu)$ of μ as $m(\mu) := \dim L^2(X, E_\chi)_\mu$.

We want to utilize the heat operator $e^{-t\Delta_\chi^\sharp}$, induced by Δ_χ^\sharp , as an integral, trace class operator and derive a corresponding trace formula. In [26], a Selberg trace formula for non-unitary twists is derived for particular test functions ϕ , the Paley-Wiener functions on \mathbb{C} (see [26, p. 2079]). Since, the exponential function $\phi(\lambda) = e^{-t\lambda^2}$, $\lambda \in \mathbb{C}$ does not belong to this space, we use the extended results obtained in [37, p. 171–173]. With this, we conclude that the heat operator $e^{-\Delta_\chi^\sharp}$ is an integral operator with smooth kernel, i.e., there exists a smooth section H_t^χ of $\text{End}(E_\chi)$ on $X \times X$ such that for $f \in L^2(X, E_\chi)$, we have

$$e^{-\Delta_\chi^\sharp} f(x) = \int_X H_t^\chi(x, y) f(y) dy.$$

By [26, Proposition 2.5], this operator is of trace class. By Lidskii’s theorem [36, Theorem 3.7], we have

$$(3.6) \quad \text{tr}(e^{-\Delta_\chi^\sharp}) = \sum_{\mu \in \text{spec}(\Delta_\chi^\sharp)} m(\mu) e^{-t\mu},$$

which is the spectral side of our (pre)-trace formula. Let H_t be the kernel of the heat operator $e^{-t\tilde{\Delta}}$, that is the operator induced by the self-adjoint Laplacian $\tilde{\Delta}$ acting in $L^2(\tilde{X})$. By [2, Lemma 2.3 and Proposition 2.4], H_t is contained

in the Harish-Chandra L^q -Schwartz space $\mathcal{C}^q(G)$ for any $q > 0$ (see e.g. [2, p. 161–162] for the definition of the Schwartz space). Moreover, as in [37, p. 174], it follows from (3.5) that

$$(3.7) \quad H_t^X(x, x') = \sum_{\gamma \in \Gamma} H_t(g^{-1}\gamma g')\chi(\gamma),$$

where $x = \Gamma gK, x' = \Gamma g'K \in X$ with $g, g' \in G$. By [37, Proposition 5.3], we have the following result.

Proposition 3.2.1. *Let E_χ be a flat vector bundle over $X = \Gamma \backslash \tilde{X}$ associated with a finite-dimensional, complex representation $\chi: \Gamma \rightarrow \mathrm{GL}(V_\chi)$ of Γ . Let Δ_χ^\sharp be the twisted Bochner–Laplace operator acting in $L^2(X, E_\chi)$. Then,*

$$(3.8) \quad \sum_{\mu \in \mathrm{Spec}(\Delta_\chi^\sharp)} m(\mu)e^{-t\mu} = \mathrm{tr}(e^{-t\Delta_\chi^\sharp}) = \int_{\Gamma \backslash G} \left(\sum_{\gamma \in \Gamma} \mathrm{tr} \chi(\gamma) \cdot H_t(g^{-1}\gamma g) \right) dg.$$

As in [26, Proposition 5.1, Proposition 6.1] and [41, p. 172–173, 177–178], we group the summation into conjugacy classes and use the Fourier inversion formula to obtain

$$(3.9) \quad \mathrm{tr}(e^{-t\Delta_\chi^\sharp}) = \dim(V_\chi) \mathrm{Vol}(X) H_t(e) + \sum_{[\gamma] \neq e} \mathrm{tr} \chi(\gamma) \frac{l(\gamma)}{n_\Gamma(\gamma) D(\gamma)} \frac{1}{2\pi} \int_{\mathbb{R}} \Theta_\lambda(H_t) e^{-it(\gamma)\lambda} d\lambda,$$

where

$$D(\gamma) = e^{-\frac{l(\gamma)}{2}} |\det(\mathrm{Ad}(a_\gamma)|_{\mathfrak{n}} - \mathrm{Id})|.$$

Note that every $\gamma \in \Gamma, \gamma \neq e$, is hyperbolic, so the summation is over hyperbolic conjugacy classes of Γ . We therefore refer to the first term in the RHS of (3.9) as the *identity contribution* and to the second term as the *hyperbolic contribution*.

Remark 3.2.2. The trace formula in [41] is obtained for kernel functions that are compactly supported which is not true for the heat kernel. However, the heat kernel is an admissible function in the sense of Gangolli [17, p. 407]. This is because the heat kernel belongs to the Harish-Chandra L^1 -Schwartz space $\mathcal{C}^1(G)$ (see [17, Proposition 3.1 and p. 411]).

Let us first simplify the hyperbolic contribution. The character Θ_λ of π_λ can be evaluated on the K -biinvariant function H_t in terms of the spherical Fourier transform $\tilde{H}_t(\lambda)$ of H_t :

$$\Theta_\lambda(H_t) = \mathrm{tr} \pi_\lambda(H_t) = \int_G \langle \pi_\lambda(g)v, v \rangle H_t(g) dg = \int_G \phi_\lambda(g) H_t(g) dg = \tilde{H}_t(\lambda),$$

where $v \in \mathcal{H}_\lambda^K$, with $\|v\| = 1$ and $\phi_\lambda(g) = \langle \pi_\lambda(g)v, v \rangle$ denotes the associated spherical function (see e.g. [20, Chapter IV] for details).

Lemma 3.2.3.

$$(3.10) \quad \Theta_\lambda(H_t) = e^{-t(\lambda^2 + \frac{1}{4})}.$$

Proof. By [2, eq. (2.11)],

$$\widetilde{H}_t(\lambda) = e^{t\pi_\lambda(\Omega)},$$

where Ω is the Casimir element in the center of the universal enveloping algebra of $\mathfrak{sl}(2, \mathbb{C})$. By [22, Proposition 8.22],

$$\pi(\Omega) = -\lambda^2 - \rho^2.$$

Recall now from Subsection 2.3 that ρ corresponds to $\frac{1}{2}$. □

We now turn to the identity contribution in (3.9). The function H_t on G belongs to the Harish-Chandra L^q -Schwartz space. Hence, the Fourier inversion formula [19, Theorem 3] can be applied to H_t (see also Theorem 1 in [1] for K -biinvariant test functions in the Harish-Chandra L^q -Schwartz space for $0 < q \leq 2$). By [20, Theorem 7.5 (i)] (see also [20, eq. (28) and (29), p. 42]), we have

$$(3.11) \quad H_t(e) = \frac{1}{4\pi^2} \int_{\mathbb{R}} \Theta_\lambda(H_t) |c(\lambda)|^{-2} d\lambda$$

with the c -function $c(\lambda)$ given by (note the change of variables λ vs. $\frac{\lambda}{2}$ compared to [20])

$$(3.12) \quad |c(\lambda)|^{-2} = \lambda\pi \tanh \lambda\pi, \quad \lambda \in \mathbb{R}.$$

Theorem 3.2.4 (Trace formula). *Let $\chi: \Gamma \rightarrow \mathrm{GL}(V_\chi)$ be a finite-dimensional representation of Γ . Then, the following Selberg trace formula for the operator $e^{-t\Delta_\chi^\sharp}$ holds:*

$$(3.13) \quad \begin{aligned} \mathrm{tr}(e^{-t\Delta_\chi^\sharp}) &= \frac{1}{4\pi^2} \dim(V_\chi) \mathrm{Vol}(X) \int_{\mathbb{R}} e^{-t(\lambda^2 + \frac{1}{4})} \lambda\pi \tanh \lambda\pi d\lambda \\ &+ \sum_{[\gamma] \neq e} \mathrm{tr} \chi(\gamma) \frac{l(\gamma)}{n_\Gamma(\gamma) D(\gamma)} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-t(\lambda^2 + \frac{1}{4})} e^{-il(\gamma)\lambda} d\lambda. \end{aligned}$$

Proof. The trace formula (3.13) follows from (3.9), (3.10), (3.11) and the explicit formula for the c -function (3.12). □

4 Twisted dynamical zeta functions on compact hyperbolic surfaces

In this section we prove meromorphic continuation of the twisted Selberg and the Ruelle zeta function, obtain their functional equations and study the behaviour of the twisted Ruelle zeta function at $s = 0$.

4.1 Definition and convergence

Let $\chi: \Gamma \rightarrow \mathrm{GL}(V_\chi)$ be a finite-dimensional, complex representation of Γ . For $s \in \mathbb{C}$, we define the twisted Selberg zeta function, associated with χ , by the infinite product

$$(4.1) \quad Z(s; \chi) := \prod_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \prod_{k=0}^{\infty} \det(\mathrm{Id} - \chi(\gamma) e^{-(s+k)l(\gamma)}).$$

Proposition 4.1.1. *There exists a positive constant c_1 such that the product (4.1) defining the twisted Selberg zeta function converges absolutely and uniformly on compact subsets of the half-plane $\operatorname{Re}(s) > c_1$.*

Proof. Let $\|\cdot\|$ be the operator norm associated to a fixed norm on V_χ . By [42, p. 10], there exists a $c \geq 0$ such that

$$(4.2) \quad \|\chi(\gamma)\| \leq e^{cl(\gamma)}.$$

Hence, for $\operatorname{Re}(s) \gg 0$, by (4.1), we get

$$\begin{aligned} \log Z(s; \chi) &= \sum_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \sum_{k=0}^{\infty} \operatorname{tr} \log(1 - \chi(\gamma)e^{-(s+k)l(\gamma)}) \\ &= - \sum_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \frac{\operatorname{tr}((\chi(\gamma)e^{-(s+k)l(\gamma)})^j)}{j} \\ &= - \sum_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \sum_{k=0}^{\infty} \frac{1}{n_\Gamma(\gamma)} \operatorname{tr}(\chi(\gamma))e^{-(s+k)l(\gamma)} \\ (4.3) \quad &= - \sum_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \frac{1}{n_\Gamma(\gamma)} \operatorname{tr}(\chi(\gamma)) \frac{e^{-sl(\gamma)}}{1 - e^{-l(\gamma)}}. \end{aligned}$$

By the inequality $|\operatorname{tr}(\chi(\gamma))| \leq \dim(V_\chi)\|\chi(\gamma)\|$ and (4.2), we get

$$(4.4) \quad |\operatorname{tr}(\chi(\gamma))| \leq c'e^{cl(\gamma)}$$

(see also [37, Lemma 3.3]). Moreover, if we define

$$\mathcal{N}_\Gamma^C(R) := \#\{[\gamma] \in C(\Gamma) : l(\gamma) \leq R\}, \quad R \geq 0,$$

where $C(\Gamma)$ denotes the set of Γ -conjugacy classes, then by [3, equation (1.31)] there exists a positive constant C such that

$$(4.5) \quad \mathcal{N}_\Gamma^C(R) \leq Ce^R.$$

Hence, the assertion follows from (4.3), (4.4) and (4.5). \square

Lemma 4.1.2. *Let $L(s; \chi) := \frac{d}{ds} \log Z(s; \chi)$ be the logarithmic derivative of $Z(s; \chi)$. Then,*

$$(4.6) \quad L(s; \chi) = \sum_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \frac{l(\gamma) \operatorname{tr}(\chi(\gamma))}{2n_\Gamma(\gamma) \sinh(l(\gamma)/2)} e^{-(s-\frac{1}{2})l(\gamma)}.$$

Proof. (4.6) follows easily by differentiating (4.3). \square

We define the twisted Ruelle zeta function, associated with χ , by the infinite product

$$(4.7) \quad R(s; \chi) = \prod_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \det(\operatorname{Id} - \chi(\gamma)e^{-sl(\gamma)}).$$

Proposition 4.1.3. *There exists a positive constant c_2 such that the product (4.7) defining the twisted Ruelle zeta function converges absolutely and uniformly on compact subsets of the half-plane $\operatorname{Re}(s) > c_2$.*

Proof. The proof is similar to the proof of Proposition 4.1.1 since

$$\begin{aligned}
\log R(s; \chi) &= \sum_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \operatorname{tr} \log(1 - \chi(\gamma)e^{-sl(\gamma)}) \\
(4.8) \qquad &= - \sum_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \sum_{j=1}^{\infty} \frac{\operatorname{tr}((\chi(\gamma)e^{-sl(\gamma)})^j)}{j} \\
&= - \sum_{[\gamma] \neq e} \frac{1}{n_{\Gamma}(\gamma)} \operatorname{tr}(\chi(\gamma))e^{-sl(\gamma)}. \quad \square
\end{aligned}$$

Remark 4.1.4. The constants c_1, c_2 in Propositions 4.1.2 and 4.1.3, respectively, can be explicitly determined. In fact, we can chose $c_1 = c_2 = c + 1$, where c is as in (4.2). We remark also that one can consider the notion of the critical exponent of χ , in order to eliminate the dependence on the norm $\|\cdot\|$ on V_{χ} in the estimates (4.2). The critical exponent c_{χ} is defined as $c_{\chi} := \inf\{c \geq 0, \text{ such that (4.2) holds}\}$. Note that for χ being unitary, we have that $c_{\chi} = 0$. For more details we refer the reader to [42, p. 10].

Lemma 4.1.5. *The twisted Ruelle zeta function and the twisted Selberg zeta function are related by the following identity:*

$$(4.9) \qquad R(s; \chi) = \frac{Z(s; \chi)}{Z(s+1; \chi)}.$$

Proof. (4.9) follows by considering (4.3) at $s+1$ and (4.8). □

4.2 Meromorphic continuation

The trace formula (3.13) motivates to consider a shift of the operator Δ_{χ}^{\sharp} by $\frac{1}{4}$:

$$A_{\chi}^{\sharp} := \Delta_{\chi}^{\sharp} - \frac{1}{4}.$$

We now explicitly calculate the hyperbolic contribution on the RHS of (3.13).

Corollary 4.2.1. *Let $X = \Gamma \backslash \mathbb{H}^2$ be a compact hyperbolic surface and χ be a finite-dimensional, complex representation of Γ . Then, we have*

$$\begin{aligned}
(4.10) \quad \operatorname{tr}(e^{-t(\Delta_{\chi}^{\sharp} - \frac{1}{4})}) &= \frac{1}{4\pi^2} \dim(V_{\chi}) \operatorname{Vol}(X) \int_{\mathbb{R}} e^{-t\lambda^2} \lambda \pi \tanh \lambda \pi d\lambda \\
&\quad + \frac{1}{2\sqrt{4\pi t}} \sum_{[\gamma] \neq e} \frac{l(\gamma) \operatorname{tr}(\chi(\gamma))}{n_{\Gamma}(\gamma) \sinh(l(\gamma)/2)} e^{-\frac{l(\gamma)^2}{4t}}.
\end{aligned}$$

Proof. We recall that

$$D(\gamma) = e^{-\frac{l(\gamma)}{2}} |\det(\operatorname{Ad}(a_{\gamma})|_{\mathfrak{n}} - \operatorname{Id})|.$$

Using

$$\det(\mathrm{Ad}(a_\gamma)|_{\mathfrak{n}} - \mathrm{Id}) = \sum_{q=0}^1 (-1)^{q+1} \mathrm{tr}(\Lambda^q \mathrm{Ad}(a_\gamma)|_{\mathfrak{n}}),$$

and [22, eq. (5.15), p. 138], we get

$$(4.11) \quad D(\gamma) = e^{-l(\gamma)/2} |(-1 + e^{l(\gamma)})| = e^{l(\gamma)/2} - e^{-l(\gamma)/2} = 2 \sinh(l(\gamma)/2).$$

Moreover, the integral in the hyperbolic contribution in the RHS of (3.13) is just the Fourier transform of the function $\lambda \mapsto e^{-t\lambda^2}$. Hence, substituting (4.11) in (3.13), we get (4.10). \square

Now let $s_1, s_2 \in \mathbb{C}$ with $(s_1 - 1/2)^2 \neq (s_2 - 1/2)^2$, such that $(s_1 - 1/2)^2, (s_2 - 1/2)^2 \in \mathbb{C} \setminus \mathrm{spec}(-A_\chi^\sharp)$. We consider the product of the two resolvents $(A_\chi^\sharp + (s_j - \frac{1}{2})^2)^{-1}$, for $j = 1, 2$. By [30], if D is an elliptic, injective differential operator, then

$$\mathrm{tr}(D^{-s}) < \infty,$$

for $\mathrm{Re}(s) > \frac{\dim(X)}{\mathrm{order}(D)}$. Hence, the product

$$(A_\chi^\sharp + (s_1 - 1/2)^2)^{-1} (A_\chi^\sharp + (s_2 - 1/2)^2)^{-1}$$

is of trace class. Now, the resolvent identity

$$(4.12) \quad \begin{aligned} & (A_\chi^\sharp + (s_1 - 1/2)^2)^{-1} (A_\chi^\sharp + (s_2 - 1/2)^2)^{-1} = \frac{1}{(s_2 - 1/2)^2 - (s_1 - 1/2)^2} \\ & \times \left((A_\chi^\sharp + (s_1 - 1/2)^2)^{-1} - (A_\chi^\sharp + (s_2 - 1/2)^2)^{-1} \right) \end{aligned}$$

implies that the difference

$$(A_\chi^\sharp + (s_1 - 1/2)^2)^{-1} - (A_\chi^\sharp + (s_2 - 1/2)^2)^{-1}$$

is also of trace class. We observe that for $\mathrm{Re}(s_j) \gg 0$

$$(A_\chi^\sharp + (s_j - 1/2)^2)^{-1} = \int_0^\infty e^{-t(s_j - 1/2)^2} e^{-tA_\chi^\sharp} dt.$$

Hence,

$$\begin{aligned} & (A_\chi^\sharp + (s_1 - 1/2)^2)^{-1} - (A_\chi^\sharp + (s_2 - 1/2)^2)^{-1} \\ & = \int_0^\infty (e^{-t(s_1 - 1/2)^2} - e^{-t(s_2 - 1/2)^2}) e^{-tA_\chi^\sharp} dt. \end{aligned}$$

Lemma 4.2.2. *There exist coefficients c_j such that $e^{-tA_\chi^\sharp}$ has an asymptotic expansion*

$$(4.13) \quad \mathrm{tr} e^{-tA_\chi^\sharp} \sim \dim(V_\chi) \sum_{j=0}^{\infty} c_j t^{\frac{j-2}{2}} \quad \text{as } t \rightarrow 0^+.$$

Proof. Since $e^{\frac{\Delta}{4}}$ is regular at $t = 0$, we may replace A_{χ}^{\sharp} in (4.13) by Δ_{χ}^{\sharp} . By (3.7), the kernel $H_t^{\chi} : X \times X \rightarrow \text{End}(E_{\chi})$ of the operator $e^{-t\Delta_{\chi}^{\sharp}}$ is given by

$$(4.14) \quad \begin{aligned} H_t^{\chi}(x, y) &= \sum_{\gamma \in \Gamma} H_t(\tilde{x}, \gamma \tilde{y}) \chi(\gamma) \\ &= H_t(\tilde{x}, \tilde{y}) \text{Id}_{V_{\tilde{x}}} + \sum_{\substack{\gamma \in \Gamma \\ \gamma \neq e}} H_t(\tilde{x}, \gamma \tilde{y}) \chi(\gamma), \end{aligned}$$

where $H_t : \tilde{X} \times \tilde{X} \rightarrow \mathbb{C}$ denotes the kernel associated with the operator $e^{-t\tilde{\Delta}}$ and $\tilde{x}, \tilde{y} \in \tilde{X}$ are lifts of $x, y \in X$. By [8, Theorem 3.3 and (P4)], for every $T > 0$, there exists a constant $C > 0$ such that for $0 < t \leq T$ and $\tilde{x}, \tilde{y} \in \tilde{X}$ we have

$$(4.15) \quad H_t(\tilde{x}, \tilde{y}) \leq Ct^{-1} e^{-\frac{d(\tilde{x}, \tilde{y})^2}{4t}}.$$

Combining (4.15) with (4.4) gives constants $c_1 > 0$ and $c_2 \geq 0$ such that

$$(4.16) \quad \sum_{\substack{\gamma \in \Gamma \\ \gamma \neq e}} |\text{tr}(\chi(\gamma))| H_t(\tilde{x}, \gamma \tilde{x}) \leq c_1 t^{-1} \sum_{\substack{\gamma \in \Gamma \\ \gamma \neq e}} e^{c_2 l(\gamma)} e^{-\frac{l(\gamma)^2}{4t}}.$$

By [16, Lemma 5.1 and Corollary 5.2] (see also [42, p. 28-30]), there exists a constant $c_0 > 0$ such that $l(\gamma) \geq c_0$ for every $\gamma \neq e$. This implies the following estimate:

$$\begin{aligned} \sum_{\substack{\gamma \in \Gamma \\ \gamma \neq e}} e^{c_2 l(\gamma)} e^{-\frac{l(\gamma)^2}{4t}} &= \sum_{k=1}^{\infty} \sum_{\substack{\gamma \neq e \\ kc_0 \leq l(\gamma) < (k+1)c_0}} e^{c_2 l(\gamma)} e^{-\frac{l(\gamma)^2}{4t}} \\ &\leq \sum_{k=1}^{\infty} \mathcal{N}_{\Gamma}((k+1)c_0) e^{c_0 c_2 (k+1)} e^{-\frac{c_0^2 k^2}{4t}}, \end{aligned}$$

where

$$\mathcal{N}_{\Gamma}(R) = \#\{\gamma \in \Gamma : l(\gamma) \leq R\} \leq c_3 e^{c_4 R}$$

for some $c_3, c_4 > 0$ by [3, equation (1.31)]. Hence, for $0 < t \leq T$ we have

$$(4.17) \quad \begin{aligned} \sum_{\substack{\gamma \in \Gamma \\ \gamma \neq e}} e^{c_2 l(\gamma)} e^{-\frac{l(\gamma)^2}{4t}} &\leq c_3 \sum_{k=1}^{\infty} e^{c_0(c_2+c_4)(k+1)} e^{-\frac{c_0^2 k^2}{4t}} \\ &\leq c_3 e^{-c_0^2/4t} \sum_{k=1}^{\infty} e^{c_0(c_2+c_4)(k+1)} e^{-\frac{c_0^2(k^2-1)}{4t}} = c_T e^{-c_0^2/4t}. \end{aligned}$$

Together with (4.16) this implies that the second term in (4.14) can be estimated by

$$(4.18) \quad \sum_{\substack{\gamma \in \Gamma \\ \gamma \neq e}} |\text{tr}(\chi(\gamma))| H_t(\tilde{x}, \gamma \tilde{x}) \leq C'' t^{-1} e^{-c_0^2/4t}$$

for some $C'' > 0$. The latter expression obviously decays exponentially as $t \rightarrow 0$. To treat the first term in (4.14), we use the classical asymptotic expansion for

the kernel H_t associated with the operator $e^{-t\tilde{\Delta}}$ on \tilde{X} (see e.g. [18, Lemma 1.7.4]):

$$H_t(\tilde{x}, \tilde{x}) \sim_{t \rightarrow 0^+} \sum_{j=0}^{\infty} c_j(\tilde{x}) t^{\frac{j-2}{2}},$$

where $c_j(\tilde{x})$ are smooth functions determined by the total symbol of the operator $\tilde{\Delta}$. Since $e^{-t\tilde{\Delta}}$ commutes with the action of G , and G acts transitively on \tilde{X} , it follows that $H_t(\tilde{x}, \tilde{x})$ is independent of \tilde{x} . Hence, the coefficients $c_j(\tilde{x})$ are also independent of \tilde{x} . Let $\tilde{x}_0 = eK \in \tilde{X}$ be the base point, then we have

$$(4.19) \quad H_t(\tilde{x}, \tilde{x}) \sim_{t \rightarrow 0^+} \sum_{j=0}^{\infty} c_j(\tilde{x}_0) t^{\frac{j-2}{2}}.$$

Multiplying with the trace of Id_{V_χ} and estimating the remaining terms in (4.14) with (4.18) shows

$$\text{tr } H_t^\chi(x, x) \sim_{t \rightarrow 0^+} \dim(V_\chi) \sum_{j=0}^{\infty} c_j t^{\frac{j-2}{2}}.$$

Finally, integrating over the compact surface X shows the claim. \square

Lemma 4.2.3. *Let $s_1, s_2 \in \mathbb{C}$ with $\text{Re}((s_1 - 1/2)^2), \text{Re}((s_2 - 1/2)^2) \gg 0$. Then,*

$$(4.20) \quad \text{tr} \left((A_\chi^\sharp + (s_1 - 1/2)^2)^{-1} - (A_\chi^\sharp + (s_2 - 1/2)^2)^{-1} \right) \\ = \int_0^\infty (e^{-t(s_1 - 1/2)^2} - e^{-t(s_2 - 1/2)^2}) \text{tr}(e^{-tA_\chi^\sharp}) dt.$$

Proof. As $t \rightarrow \infty$, the integrand in the RHS of (4.20), decays exponentially. As $t \rightarrow 0$, we use Lemma 4.2.2 and the Taylor series expansion of $e^{-t(s_1 - 1/2)^2}$ and $e^{-t(s_2 - 1/2)^2}$ to conclude that the integral converges absolutely. Hence, we can interchange summation and integration and (4.20) follows. \square

Proposition 4.2.4 (Resolvent trace formula). *Let $s_1, s_2 \in \mathbb{C}$ with $\text{Re}((s_1 - 1/2)^2), \text{Re}((s_2 - 1/2)^2) \gg 0$. Then,*

$$(4.21) \quad \text{tr} \left((A_\chi^\sharp + (s_1 - 1/2)^2)^{-1} - (A_\chi^\sharp + (s_2 - 1/2)^2)^{-1} \right) \\ = \left(\frac{1}{4\pi^2} \dim(V_\chi) \text{Vol}(X) \right) \left(\int_{\mathbb{R}} \frac{\lambda \pi \tanh \lambda \pi}{(s_1 - \frac{1}{2})^2 + \lambda^2} - \frac{\lambda \pi \tanh \lambda \pi}{(s_2 - \frac{1}{2})^2 + \lambda^2} d\lambda \right) \\ + \frac{1}{2(s_1 - \frac{1}{2})} L(s_1; \chi) - \frac{1}{2(s_2 - \frac{1}{2})} L(s_2; \chi).$$

Proof. We substitute $\text{tr}(e^{-tA_\chi^\sharp})$ in the RHS of (4.20) with the RHS of (4.10). Then, the RHS of (4.20) is equal to

$$(4.22) \quad \text{RHS}((4.20)) = I(s_1, s_2; \chi) + H(s_1, s_2; \chi),$$

where

$$I(s_1, s_2; \chi) = \frac{1}{4\pi^2} \dim(V_\chi) \text{Vol}(X) \int_0^\infty \left(e^{-t(s_1-1/2)^2} - e^{-t(s_2-1/2)^2} \right) \times \left(\int_{\mathbb{R}} e^{-t\lambda^2} \lambda \pi \tanh \lambda \pi d\lambda \right) dt$$

and

$$H(s_1, s_2; \chi) = \int_0^\infty \frac{1}{2\sqrt{4\pi t}} \left(e^{-t(s_1-1/2)^2} - e^{-t(s_2-1/2)^2} \right) \times \left(\sum_{[\gamma] \neq e} \frac{l(\gamma) \text{tr}(\chi(\gamma))}{n_\Gamma(\gamma) \sinh(l(\gamma)/2)} e^{-\frac{l(\gamma)^2}{4t}} \right) dt.$$

We first compute $I(s_1, s_2; \chi)$. Substituting $\lambda' = \lambda\sqrt{t}$ and using $|\tanh(x)| \leq 1$, we find

$$\begin{aligned} \int_{\mathbb{R}} \left| \left(e^{-t(s_1-1/2)^2} - e^{-t(s_2-1/2)^2} \right) e^{-t\lambda^2} \lambda \pi \tanh \lambda \pi \right| d\lambda \\ = \left| \frac{e^{-t(s_1-1/2)^2} - e^{-t(s_2-1/2)^2}}{t} \right| \pi \int_0^\infty \lambda' e^{-(\lambda')^2} d\lambda', \end{aligned}$$

where the latter integral over λ' is finite and independent of t and the factor in front is integrable over $t \in (0, \infty)$ by the Taylor expansion of $e^{-t(s_j-1/2)^2}$. Hence, we can interchange the integrals in the expression of $I(s_1, s_2; \chi)$ and compute the inner integral over t to obtain

$$(4.23) \quad I(s_1, s_2; \chi) = \frac{1}{4\pi^2} \dim(V_\chi) \text{Vol}(X) \times \int_{\mathbb{R}} \left(\frac{\lambda \pi \tanh \lambda \pi}{(s_1 - \frac{1}{2})^2 + \lambda^2} - \frac{\lambda \pi \tanh \lambda \pi}{(s_2 - \frac{1}{2})^2 + \lambda^2} \right) d\lambda.$$

For computing $H(s_1, s_2; \chi)$, we use

$$\int_0^\infty \frac{1}{\sqrt{4\pi t}} e^{-t(s-\frac{1}{2})^2} e^{-\frac{l(\gamma)^2}{4t}} dt = \frac{1}{2(s-\frac{1}{2})} e^{-(s-\frac{1}{2})l(\gamma)},$$

for $\text{Re}((s - \frac{1}{2})^2) > 0$ ([11, (27), p. 146]). Hence,

$$\begin{aligned} H(s_1, s_2; \chi) = & \left(\frac{1}{2(s_1 - \frac{1}{2})} \sum_{[\gamma] \neq e} \frac{l(\gamma) \text{tr}(\chi(\gamma))}{2n_\Gamma(\gamma) \sinh(l(\gamma)/2)} e^{-(s_1 - \frac{1}{2})l(\gamma)} \right. \\ & \left. - \frac{1}{2(s_2 - \frac{1}{2})} \sum_{[\gamma] \neq e} \frac{l(\gamma) \text{tr}(\chi(\gamma))}{2n_\Gamma(\gamma) \sinh(l(\gamma)/2)} e^{-(s_2 - \frac{1}{2})l(\gamma)} \right). \end{aligned}$$

By (4.6), we get

$$(4.24) \quad H(s_1, s_2; \chi) = \frac{1}{2(s_1 - \frac{1}{2})} L(s_1; \chi) - \frac{1}{2(s_2 - \frac{1}{2})} L(s_2; \chi).$$

The claim now follows by putting together (4.22), (4.23) and (4.24). \square

Proposition 4.2.5. *The logarithmic derivative $L(s; \chi)$ of the Selberg zeta function $Z(s; \chi)$ extends to a meromorphic function in $s \in \mathbb{C}$ with singularities given by the following formal expression:*

$$(4.25) \quad \sum_{j=0}^{\infty} \left[\frac{1}{s - \frac{1}{2} - i\mu_j} + \frac{1}{s - \frac{1}{2} + i\mu_j} \right] + \frac{\text{Vol}(X) \dim(V_\chi)}{2\pi} \sum_{k=0}^{\infty} \frac{1+2k}{s+k}.$$

where $(\lambda_j = \frac{1}{4} + \mu_j^2)_{j \in \mathbb{Z}^+} \subseteq \mathbb{C}$ are the eigenvalues of Δ_χ^\sharp counted with algebraic multiplicity.

Proof. We fix $s_2 \in \mathbb{C}$ with $\text{Re}((s_2 - \frac{1}{2})^2) \gg 0$ and let $s = s_1 \in \mathbb{C}$ vary. Multiplying both sides of (4.21) with $2(s - \frac{1}{2})$ we see that the LHS of (4.21), that is the spectral side of the resolvent trace formula, gives exactly the singularities of the first term in (4.25).

In order to obtain the singularities of the second term in (4.25), we recall the expression (4.23) and use the following identity (see e.g. [22, p. 401]):

$$\pi \lambda \tanh \pi \lambda = \sum_{\substack{n=-\infty \\ n=\text{odd}}}^{+\infty} \frac{\lambda^2}{(\frac{n}{2})^2 + \lambda^2}.$$

Then, we get

$$(4.26) \quad 2\left(s - \frac{1}{2}\right) I(s, s_2; \chi) = \left(\frac{1}{4\pi^2} \dim(V_\chi) \text{Vol}(X) \right) \cdot \left(2\left(s - \frac{1}{2}\right) \right) \\ \times \left[\int_{\mathbb{R}} \left(\frac{1}{(s - \frac{1}{2})^2 + \lambda^2} - \frac{1}{(s_2 - \frac{1}{2})^2 + \lambda^2} \right) \left(\sum_{\substack{n=-\infty \\ n=\text{odd}}}^{+\infty} \frac{\lambda^2}{(\frac{n}{2})^2 + \lambda^2} \right) d\lambda \right].$$

Since one can interchange summation and integration on the RHS of the equation above, we obtain the following expression

$$I := \sum_{\substack{n=-\infty \\ n=\text{odd}}}^{+\infty} \left(\int_{\mathbb{R}} \frac{\lambda^2}{((s - \frac{1}{2})^2 + \lambda^2)((\frac{n}{2})^2 + \lambda^2)} - \frac{\lambda^2}{((s_2 - \frac{1}{2})^2 + \lambda^2)((\frac{n}{2})^2 + \lambda^2)} d\lambda \right).$$

To compute the integral, note that for $a^2 \neq b^2$ we can write

$$\frac{\lambda^2}{(a^2 + \lambda^2)(b^2 + \lambda^2)} = \frac{1}{a^2 - b^2} \left(\frac{a^2}{a^2 + \lambda^2} - \frac{b^2}{b^2 + \lambda^2} \right)$$

and use the following integral formula, which is easily derived using the residue theorem:

$$\int_{\mathbb{R}} \frac{1}{a^2 + \lambda^2} d\lambda = \frac{\pi}{a} \quad \text{for } \text{Re}(a) > 0.$$

This yields

$$I = \sum_{\substack{n=-\infty \\ n=\text{odd}}}^{+\infty} \left(\frac{\pi}{s + \frac{|n|-1}{2}} - \frac{\pi}{s_2 + \frac{|n|-1}{2}} \right) = \sum_{\substack{n=-\infty \\ n=\text{odd}}}^{+\infty} \frac{\pi(s_2 - s)}{(s + \frac{|n|-1}{2})(s_2 + \frac{|n|-1}{2})},$$

and hence, (4.26) gives

$$(4.27) \quad 2\left(s - \frac{1}{2}\right) I(s; s_2, \chi) = \frac{1}{2\pi} \dim(V_\chi) \operatorname{Vol}(X) \sum_{\substack{n=-\infty \\ n=\text{odd}}}^{+\infty} \frac{(s_2 - s)(s - \frac{1}{2})}{(s + \frac{|n|-1}{2})(s_2 + \frac{|n|-1}{2})}.$$

The result now follows by substituting $k = \frac{|n|-1}{2} \in \mathbb{Z}^+$. \square

Theorem 4.2.6. *Let $X = \Gamma \backslash \mathbb{H}^2$ be a compact hyperbolic surface and let $\chi: \Gamma \rightarrow \operatorname{GL}(V_\chi)$ be a finite-dimensional, complex representation of Γ . Then, the twisted Selberg zeta function $Z(s; \chi)$ admits a meromorphic continuation to \mathbb{C} . The zeros/singularities are contained in the set $\{\frac{1}{2} \pm i\mu_j : j \in \mathbb{Z}^+\} \cup \mathbb{Z}^-$.*

Proof. The theorem follows from Proposition 4.2.5 by integration and exponentiation. We only need to prove that the residues of $L(s; \chi)$ are integers. This follows from (4.25) and the Gauss–Bonnet formula

$$\frac{\operatorname{Vol}(X)}{2\pi} = 2g - 2,$$

where g is the genus of the surface. \square

Remark 4.2.7. Since the twisted Bochner–Laplace operator Δ_χ^\sharp is in general not self-adjoint, the eigenvalues $\lambda_j = \frac{1}{4} + \mu_j^2$ need not be real and > 0 and hence, some of the singularities $\frac{1}{2} + i\mu_j$ and $-k \in \mathbb{Z}^-$ of $L(s; \chi)$ could coincide. Therefore, it is not possible in general to deduce more precise information about the poles and zeros of $Z(s; \chi)$ from Proposition 4.2.5.

Corollary 4.2.8. *The twisted Ruelle zeta function $R(s; \chi)$ admits a meromorphic continuation to \mathbb{C} .*

Proof. The statement follows from Theorem 4.2.6 and Lemma 4.1.5. \square

Theorem 4.2.9. *The twisted Selberg zeta function satisfies the following functional equation.*

$$(4.28) \quad \eta(s; \chi) = \frac{Z(s; \chi)}{Z(1-s; \chi)} = \exp \left[\dim(V_\chi) \operatorname{Vol}(X) \int_0^{s-\frac{1}{2}} r \tan \pi r \, dr \right].$$

Proof. Write again $s = s_1$ and consider the transform $s \mapsto 1-s$ in (4.21). Then, the LHS of (4.21) remains invariant under this transform. The RHS of (4.21) is as in (4.22), where $I(s, s_2; \chi)$ is now given by (4.26) and $H(s, s_2; \chi)$ is given by (4.24). Thus, if we consider the transform $s \mapsto 1-s$, then the RHS of (4.21) will give

$$(4.29) \quad I(1-s; s_2, \chi) = \frac{1}{4\pi} \dim(V_\chi) \operatorname{Vol}(X) \sum_{\substack{n=-\infty \\ n=\text{odd}}}^{+\infty} \frac{s + s_2 - 1}{(1-s + \frac{|n|-1}{2})(s_2 + \frac{|n|-1}{2})}$$

and

$$(4.30) \quad H(1-s; s_2, \chi) = -\frac{1}{2(s - \frac{1}{2})} L(1-s; \chi) - \frac{1}{2(s_2 - \frac{1}{2})} L(s_2; \chi).$$

Hence, by the above observation, (4.29) and (4.30), we have

$$H(s; s_2, \chi) - H(1 - s; s_2, \chi) = -I(s; s_2, \chi) + I(1 - s; s_2, \chi),$$

which further gives

$$\begin{aligned} \frac{L(s; \chi) + L(1 - s; \chi)}{2(s - \frac{1}{2})} &= \frac{1}{4\pi} \dim(V_\chi) \text{Vol}(X) \sum_{\substack{n=-\infty \\ n=\text{odd}}}^{+\infty} \frac{2s - 1}{(s + \frac{|n|-1}{2})(1 - s + \frac{|n|-1}{2})} \\ &= \frac{1}{2\pi} \dim(V_\chi) \text{Vol}(X) \sum_{k=0}^{\infty} \frac{2s - 1}{(s + k)(1 - s + k)} \\ &= \frac{1}{2\pi} \dim(V_\chi) \text{Vol}(X) \sum_{k=0}^{\infty} \left(\frac{1}{1 - s + k} - \frac{1}{s + k} \right). \end{aligned}$$

Using the classical identity

$$\pi \tan \pi x = 2x \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1}{2})^2 - x^2} = \sum_{k=0}^{\infty} \left(\frac{1}{k + \frac{1}{2} - x} - \frac{1}{k + \frac{1}{2} + x} \right)$$

this can be rewritten as

$$(4.31) \quad L(s; \chi) + L(1 - s; \chi) = \dim(V_\chi) \text{Vol}(X) \left(s - \frac{1}{2} \right) \tan \pi \left(s - \frac{1}{2} \right).$$

We integrate both sides of (4.31) over s and exponentiate the result to get (4.28). \square

Corollary 4.2.10. *The twisted Ruelle zeta function satisfies the following functional equation*

$$(4.32) \quad R(s; \chi)R(-s; \chi) = \exp \left[-\dim(V_\chi) \text{Vol}(X) \int_{s-\frac{1}{2}}^{s+\frac{1}{2}} r \tan \pi r \, dr \right].$$

Proof. By Lemma 4.1.5, we have

$$(4.33) \quad R(s; \chi)R(-s; \chi) = \frac{Z(s; \chi)}{Z(1 + s; \chi)} \frac{Z(-s; \chi)}{Z(1 - s; \chi)}.$$

Applying (4.28) for s and $-s$ shows the claimed formula. \square

4.3 Behaviour of the twisted Ruelle zeta function at zero

Since it is not possible to obtain precise information on the behaviour of $Z(s; \chi)$ at $s = 0$ and $s = 1$ from Proposition 4.2.5, we cannot directly use (4.9) to obtain the behaviour of $R(s; \chi)$ at $s = 0$. Instead, we make use of the functional equation for $Z(s; \chi)$:

Corollary 4.3.1. *The twisted Ruelle zeta function $R(s; \chi)$ near $s = 0$ is given by*

$$(4.34) \quad R(s; \chi) = \pm (2\pi s)^{\dim(V_\chi)(2g-2)} + \text{higher order terms.}$$

Proof. By Lemma 4.1.5 and (4.28), we have as $s \rightarrow 0$:

$$(4.35) \quad R(s; \chi) = \frac{Z(s; \chi)}{Z(s+1; \chi)} = \frac{Z(s; \chi)}{\eta(s+1)Z(-s; \chi)} \sim \pm \frac{1}{\eta(s+1; \chi)}.$$

Hence, by (4.28), we get

$$(4.36) \quad \eta(s+1; \chi)^{-1} = \exp \left[-\dim(V_\chi) \text{Vol}(X) \int_0^{s+\frac{1}{2}} r \tan \pi r \, dr \right].$$

We set

$$(4.37) \quad A(s) = -2\pi \int_0^{s+\frac{1}{2}} r \tan \pi r \, dr.$$

Let $s = -\epsilon$, where ϵ is a positive real number. Then,

$$(4.38) \quad \begin{aligned} A(-\epsilon) &= -2\pi \int_0^{\frac{1}{2}-\epsilon} r \tan(\pi r) \, dr \\ &= -2\pi \int_0^{\frac{1}{2}-\epsilon} r \frac{\sin(2\pi r)}{1 + \cos(2\pi r)} \, dr \\ &= [r \log(1 + \cos(2\pi r))]_{r=0}^{r=\frac{1}{2}-\epsilon} - \int_0^{\frac{1}{2}-\epsilon} \log(1 + \cos(2\pi r)) \, dr. \end{aligned}$$

By [15, p.153], the first term in (4.38) is given by

$$\begin{aligned} [r \log(1 + \cos(2\pi r))]_{r=0}^{r=\frac{1}{2}-\epsilon} &= \left(\frac{1}{2} - \epsilon \right) \log \left(1 + \cos \left(2\pi \left(\frac{1}{2} - \epsilon \right) \right) \right) \\ &= \left(\frac{1}{2} - \epsilon \right) \log (2\pi^2 \epsilon^2) (1 + O(\epsilon)). \end{aligned}$$

Moreover, by [15, p.153], the latter integral in (4.38) is of order $\frac{1}{2} \log \frac{1}{2} + O(\epsilon)$. Hence, as $\epsilon \rightarrow 0$

$$(4.39) \quad \exp(A(-\epsilon)) \sim 2\pi\epsilon = -2\pi s.$$

The assertion follows by (4.35), (4.36), (4.37) and (4.39). \square

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