The twisted Ruelle zeta function on compact hyperbolic orbisurfaces and Reidemeister–Turaev torsion

Léo Bénard, Jan Frahm, Polyxeni Spilioti

October 14, 2021

Abstract

Let X be a compact hyperbolic surface with finite order singularities, X_1 its unit tangent bundle. We consider the Ruelle zeta function $R(s;\rho)$ associated to a representation $\rho\colon \pi_1(X_1) \to \mathrm{GL}(V_\rho)$. If ρ does not factor through $\pi_1(X)$, we show that the value at 0 of the Ruelle zeta function is the Reidemeister–Turaev torsion of (X_1, ρ) with respect to the Euler structure induced by the geodesic flow, thus generalizing Fried's conjecture to this setting. We also compute the vanishing order and the leading coefficient of the Ruelle zeta function at $s = 0$ when ρ factors through $\pi_1(X)$.

Keywords — hyperbolic orbisurface, twisted Ruelle zeta function, non-unitary representation, Reidemeister–Turaev torsion, Selberg trace formula.
Mathematics Subject Classification (2020) — Primary: 11M36; Secondary: 11F72, 37C30, 57Q10.

Introduction

Given a compact Riemannian manifold M, a representation $\rho\colon \pi_1(M) \to \mathrm{GL}(V_\rho)$ and a vector field $\mathfrak{X}$ on M, the Ruelle zeta function is defined as

$$R(s;\rho) = \prod_{\gamma \text{ prime}} \det\left(\mathrm{Id} - \rho(\gamma) e^{-s\ell(\gamma)}\right),$$

where γ runs through the *prime* periodic orbits of $\mathfrak{X}$ and $\ell(\gamma)$ denote the length of the orbit γ.

In this work we are interested in the case where $M = X_1$ is the unit tangent bundle of a compact hyperbolic surface X with finitely many singular points $x_1, \dots x_s$ of finite order. The geodesic flow $\mathfrak{X}$ acts on X_1 and its (prime) periodic orbits are lifts of (prime) closed geodesics on the surface. For each singular point x_j on X there is a class of loop c_j encircling x_j. It lifts with finite order ν_j to $\pi_1(X_1)$. We also denote by t the class of the generic fiber. It represents a loop in X_1 of a unit vector which makes a full positive rotation around the base point.

Let $\rho\colon \pi_1(X_1) \to \mathrm{GL}(V_\rho)$ be a representation with $\dim V_\rho = n$. For any $j = 1, \dots, s$ we denote by $n_j = \dim \operatorname{Fix} \rho(c_j)$, so that $\rho(c_j) = I_{n_j} \oplus T_j$. Let $M_j(x) = \det(I_{n-n_j} - T_j)(\frac{ix}{\nu_j})^{n_j}$ and $M_0(x) = (ix)^n$ for $x \in \mathbb{C}$.

The main result of this article is the computation of the behavior at zero of the Ruelle zeta function associated to ρ.

Theorem A (Theorem 2.3.1 and Theorem 4.3.2)**.** *For any irreducible representation $\rho\colon \pi_1(X_1) \to \mathrm{GL}(V_\rho)$, the Ruelle zeta function $R(s;\rho)$ converges on some right half plane in $\mathbb{C}$ and extends meromorphically to the whole complex plane. Moreover:*

1. *If $\rho(t) = \mathrm{Id}_{V_\rho}$, then $R(s;\rho)$ vanishes at $s = 0$ with order and leading coefficient prescribed by:*
$$R\left(\frac{x}{2\pi},\rho\right) \sim_{x\to 0} \pm\frac{M_0(x)^{2g-2+s}}{\prod_{j=1}^{s} M_j(x)}.$$
2. *If $\rho(t) \neq \mathrm{Id}_{V_\rho}$, then the representation ρ is acyclic. Let $\mathfrak{e}_{geod}$ be the Euler structure induced by the geodesic flow on X_1. Then*
$$R(0;\rho) = \pm\,\mathrm{tor}(X_1, V_\rho, \mathfrak{e}_{geod})$$
where $\mathrm{tor}(X_1, V_\rho, \mathfrak{e}) \in \mathbb{C}^\times$ denotes the Reidemeister–Turaev torsion of X_1 in the representation V_ρ and the Euler structure $\mathfrak{e}_{geod}$ (see below for details).

When the representation ρ satisfies $\rho(t) \neq \mathrm{Id}_{V_\rho}$ we prove that the twisted homology groups $H_*(X_1, V_\rho)$ are trivial (see Section 2) and one can define a combinatorial invariant, the Reidemeister–Turaev torsion. A theorem of Chapman (see [3, Theorem 1]) states that it is indeed a topological invariant, although it depends in general on the choice of an Euler structure (see [27]), that is, a choice of a lift of each cell of a cell decomposition of X_1 to its universal cover $\widetilde{X_1}$. For the unit tangent bundle X_1, there is a natural choice of an Euler structure $\mathfrak{e}_{geod}$ induced by the geodesic flow and we compute explicitly in Section 2.3 the Reidemeister–Turaev torsion $\mathrm{tor}(X_1, \rho, \mathfrak{e}_{geod}) \in \mathbb{C}$ for this Euler structure. Theorem A (2) says that it coincides with the value at zero of the Ruelle zeta function, up to sign.

In the case where the representation ρ is unitary, the absolute value of the Reidemeister torsion is well-defined, independently from an Euler structure. In this setting Theorem A generalizes [10, Theorem 2 and 3], where Fried proved a similar statement for unitary and *classical* representation. He claimed without a proof that his results still hold true for general (non-classical) unitary representation. Our work fills this gap and extends Fried's results to non-unitary representations. Even in the unitary case, we drop the phase ambiguity for the torsion using the accurate Euler structure and show that it equals the value of the Ruelle zeta function at zero. The sign ambiguity for the torsion could also be removed, but not in our computation of the value $R(0;\rho)$.

Note that the value $s = 0$ is not in the convergence domain of the infinite product $R(s;\rho)$, but as we mentioned above the Ruelle zeta function extends meromorphically. On the other hand, this function strongly depends on the metric we fixed on the orbifold X and on the vector field acting on X_1. A striking consequence of Theorem A (2) is that, for a given hyperbolic metric on X and for the associated geodesic flow, the value at zero of the Ruelle zeta function is independent of these choices.

In the case the surface has no elliptic points, Theorem A (1) recovers [8, Corollary C]. Note that Theorem A (2) implies that both the Ruelle at 0 and the torsion are locally constant on the representation variety, and invariant under the action of the mapping class group.

Relation to previous results. In a more general setting, metric independence of the value of Ruelle zeta functions at 0, and its relation with Reidemeister torsion, has been widely studied in the past almost 40 years. For ρ unitary, it is known as *Fried's conjecture* and has been settled as a theorem for compact, hyperbolic odd-dimensional manifolds by Fried in [9], using the Selberg trace formula. Bunke and Olbrich in [2] unified the treatment for the twisted dynamical zeta functions with unitary representations, for all locally symmetric spaces of real rank one, using the Selberg trace formula and tools for harmonic analysis on locally symmetric spaces. The higher rank case was treated by Moscovici and Stanton in [16], and Shen in [22]. A similar result, but in variable curvature for 3-dimensional manifolds, has been proved by Dang–Guillarmou–Rivière–Shen in [5], using completely different techniques.

In the case of non-unitary representations ρ, fewer is currently known, despite it is what occurs in many interesting cases, such as holonomy representations of hyperbolic manifolds. If ρ is obtained as the restriction of a representation of a Lie group G, then Fried's conjecture has been established for $G = \mathrm{SL}_2(\mathbb{C})$ by Müller [18] (using the PhD work of Wotzke [29]) and generalized to arbitrary G by Shen [24]. Moreover, Müller ([19]), Shen ([23]) and the third author ([25]) proved Fried's conjecture for more general non-unitary representations, with the additional assumption that they are close to a unitary and acyclic representation in the representation variety. In a completely general setting, Chaubet–Dang established in [4] a variational formula relating the value at zero of the Ruelle zeta function to the Reidemeister–Turaev torsion. As a consequence of their work together with our Theorem A we have the following corollary:

Corollary B. *The statements in Theorem A still hold true on an open neighborhood in $C^\infty(X_1, TX_1)$ of the geodesic flow $\mathfrak{X}$ of any hyperbolic metric on X.*

In the present paper, the representation ρ is neither required to be unitary, nor does it have to be close to a unitary acyclic representation. We even allow representations for which the non-self adjoint Laplacian might have some generalized 0-eigenvalues. This is because we do not use the relation between the Selberg zeta function and the refined analytic torsion. Indeed, we compute directly the value of the Ruelle zeta function at zero, and identify it with the Reidemeister torsion.

Recently, Yamaguchi [30] was able to show Theorem A (2) in the special case where $\rho = \rho_{2N}$ is the restriction of the irreducible $2N$-dimensional representation of $\mathrm{SL}(2, \mathbb{R})$ to $\pi_1(X_1)$. His proof uses the relation

$$R(s; \rho_{2N}) = \frac{Z(s - N + \frac{1}{2})}{Z(s + N + \frac{1}{2})}$$

between the twisted Ruelle zeta function $R(s; \rho_{2N})$ and the non-twisted Selberg zeta function $Z(s)$ (associated with the trivial representation of $\pi_1(X_1)$) and hence does not generalize to arbitrary representations ρ of $\pi_1(X_1)$.

Outline of the proof. Our proof uses the intimate relation between the twisted Ruelle zeta function $R(s;\rho)$ and the twisted Selberg zeta function $Z(s;\rho)$ (see Section 4.1):

$$R(s;\rho) = \frac{Z(s;\rho)}{Z(s+1;\rho)}.$$

The behavior of $Z(s;\rho)$ at $s=0$ and $s=1$ can be studied through its functional equation relating $Z(s;\rho)$ and $Z(1-s;\rho)$ which we derive in Theorem 4.2.1. Both the meromorphic continuation and the functional equation of $Z(s;\rho)$ are obtained from a twisted Selberg trace formula (see Theorem 3.7.1). This trace formula arises from the trace of the heat operator of Müller's twisted Laplacian $\Delta^{\sharp}_{\tau,\rho}$ on X acting on sections of a certain vector bundle $E_{\tau,\rho}$ over X (see Müller [17, Section 4] for the definition of the twisted connexion Laplacian operator). The vector bundle $E_{\tau,\rho}$ over X is associated with the representation ρ of $\pi_1(X_1)$ and a character τ of the universal covering group of $\mathrm{PSO}(2)$. The twist by the character τ is necessary in order to obtain a vector bundle over X (see Section 3.1 for details). We note that τ can only be trivial in the case where ρ factors through $\pi_1(X)$ and hence this twisted construction is crucial for Theorem A (2).

The technical part of the proof consists of a detailed analysis of the identity, hyperbolic and elliptic contribution to the geometric side of the trace formula. This generalizes Hejhal's trace formula [12, Chapter 9, Theorem 6.2] to the case of non-unitary representations ρ and uses Hoffmann's computations of orbital integrals for the universal covering group $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$ of $\mathrm{PSL}(2,\mathbb{R})$. The main difficulty is the use of the non-self adjoint operator $\Delta^{\sharp}_{\tau,\rho}$. This operator is not necessarily diagonalizable and we have to deal with generalized eigenspaces. Moreover, the eigenvalues of this operator are not necessarily real and positive numbers but complex (see Section 3.2 for further details). We remark that the generalized eigenfunctions of $\Delta^{\sharp}_{\tau,\rho}$ can be viewed as automorphic forms on $\mathbb{H}^2$ of weight m with values in the non-unitary representation ρ (see [12, Chapter 9], and [13] for the unitary case).

On the torsion side, the very simple combinatorial nature of the unitary tangent bundle allows to compute explicitly the Euler structure $\mathfrak{e}_{geod}$ induced by the geodesic flow, and the Reidemeister–Turaev torsion in this Euler structure. It relies on Turaev's correspondence between *combinatorial* Euler structures, which yield a choice of lifts of a cell decomposition of X_1 to its universal cover necessary to compute the torsion, and *smooth* Euler structures, given by non-vanishing vector fields on X_1. To our best knowledge, it is the only case this computation has been performed explicitly, since this correspondence is not very explicit in the direction we use. Namely, given a non-vanishing vector field on a 3-manifold M, it looks in general like a difficult task to find the corresponding combinatorial Euler structure.

Finally, the last part of the proof consists of an explicit computation of the Ruelle zeta function at $s=0$. It follows the ideas of [10, Section 3], where Fried computed the modulus of each term contributing to $R(0;\rho)$ from the functional equation of the Selberg zeta function, in the case ρ was unitary. In this case the Ruelle zeta function is real for s real, hence he argued that he could forget the arguments of the terms occurring in the computation. Of course, in our case where ρ is no longer unitary, these are exactly what we have to compute, and it turns out that after tedious computations (Section 4.3) they kind of miraculously cancel out with each other.

Organization of the paper. The paper is organized as follows. In Section 1, we introduce our geometric setting, recall some facts from the representation theory of $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$ and fix notation. In Section 2, we define and compute the Reidemeister–Turaev torsion of the unit tangent bundle X_1 of the hyperbolic orbifold X in the Euler structure induced by the geodesic flow. In Section 3, we establish the Selberg trace formula for the twisted, non-self adjoint Laplacian on X, which will be the key point to prove Theorem A. In Section 4, we introduce the twisted Ruelle and Selberg zeta functions and prove their meromorphic continuation to the whole complex plane. Finally, we compute the value of the Ruelle zeta function at zero and prove Theorem A.

Acknowledgments. The authors would like to thank Adrien Boulanger, François Costantino, Nguyen Viet Dang, Pierre Dehornoy, Werner Müller, Joan Porti and Louis-Hadrien Robert for helpful discussions and comments. The first author is partially funded by the Research Training Group 2491 "Fourier Analysis and Spectral Theory", University of Göttingen. The second and third author were supported by a research grant from the Villum Foundation (Grant No. 00025373).

1 Preliminaries

We collect some basic information about compact hyperbolic orbisurfaces X (Section 1.1), their unit tangent bundles X_1 and their fundamental groups $\pi_1(X_1)$ (Section 1.2), and recall some representation theory of the universal cover of $\mathrm{PSL}(2,\mathbb{R})$ (Section 1.3).

1.1 Compact hyperbolic orbisurfaces

We consider the upper half plane

$$\mathbb{H}^2 := \{z = x + iy : y > 0\}.$$

The group $G = \mathrm{PSL}(2,\mathbb{R})$ acts on $\mathbb{H}^2$ by fractional linear transformations. This action is transitive and the stabilizer of $i \in \mathbb{H}^2$ is the maximal compact subgroup $K = \mathrm{PSO}(2)$, hence

$$\mathbb{H}^2 \cong G/K.$$

Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ be the Lie algebra of G and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to the maximal compact subgroup K. The G-invariant metric on $\mathbb{H}^2$ induced by the restriction of the Killing form on $\mathfrak{g}$ to $\mathfrak{p}$ is the Poincaré metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

Let $\Gamma \subseteq G$ be a cocompact Fuchsian group, i.e., a discrete subgroup of G such that the quotient $X = \Gamma\backslash\mathbb{H}^2 = \Gamma\backslash G/K$ is compact. Then, the Poincaré metric on $\mathbb{H}^2$ induces a metric on X that turns it into a compact hyperbolic orbisurface. Note that in this case, every non-trivial element in Γ is either hyperbolic or elliptic.

Let $\widetilde{G}$ denote the universal cover of G and write $\widetilde{H}$ for the preimage of a subgroup $H \subseteq G$ under the universal covering map. Consider the following one-parameter families in G (modulo $\pm I_2$):

$$k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad a_t = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}, \qquad n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},$$

where $\theta, t, x \in \mathbb{R}$. Denote by $\widetilde{k}_\theta$, $\widetilde{a}_t$ and $\widetilde{n}_x$ the unique lifts to the universal cover $\widetilde{G}$, turning

$$\widetilde{K} = \{\widetilde{k}_\theta : \theta \in \mathbb{R}\}, \qquad \widetilde{A} = \{\widetilde{a}_t : t \in \mathbb{R}\} \qquad \text{and} \qquad \widetilde{N} = \{\widetilde{n}_x : x \in \mathbb{R}\}$$

into connected one-parameter subgroups of $\widetilde{G}$. Then the Iwasawa decomposition

$$\widetilde{G} = \widetilde{N}\widetilde{A}\widetilde{K}$$

holds. Note that the center $\widetilde{Z}$ of $\widetilde{G}$ is given by

$$\widetilde{Z} = \{\widetilde{k}_\theta : \theta \in \mathbb{Z}\pi\}.$$

1.2 The relation between $\pi_1(X_1)$ and $\pi_1(X)$.

The aim of this section is to shed light on the relations between the fundamental group of the orbifold surface X and the fundamental group of its unit tangent bundle X_1. For this, we follow [10, Section 2].

We first note that the unit tangent bundle of $\mathbb{H}^2$ naturally identifies with $G = \mathrm{PSL}(2, \mathbb{R})$ by the map

$$\begin{aligned} G &\to \mathbb{H}^2 \\ g &\mapsto g \cdot \mathrm{i}. \end{aligned}$$

It follows from the preceding discussion that the quotient $\Gamma \backslash G$ is a compact 3-manifold X_1, which naturally identifies with the unit tangent bundle of X. This fibration induces an exact sequence

$$1 \to \mathbb{Z} = \pi_1(\mathrm{PSO}(2)) \to \pi_1(X_1) \to \Gamma = \pi_1(X) \to 1.$$

In fact, a compact orbisurface X must have finitely many conical points $x_1, \dots x_s$. Removing small disc neighborhoods $D_1, \dots D_s$ of those points, one gets a compact surface $\overline{X}$ with boundary $\partial\overline{X} = \{\partial D_1, \dots \partial D_s\}$ a union of disjoints circles, and with fundamental group

$$\pi_1(\overline{X}) = \langle a_1, b_1, \dots, a_g, b_g, c_1, \dots c_s \mid \prod_i [a_i, b_i] \prod_j c_j = 1 \rangle.$$

Capping off the boundary components ∂D_i by orbidiscs, one gets the following presentation for the fundamental group of the orbisurface:

$$\pi_1(X) = \langle a_1, b_1, \dots, a_g, b_g, c_1, \dots c_s \mid \prod_i [a_i, b_i] \prod_j c_j = 1,\ c_j^{\nu_j} = 1 \rangle,$$

for some non-zero natural integers ν_j, $i = 1, \dots s$. Those integers are the order of the elliptic conjugacy classes in Γ corresponding to the singular points x_i.

To compute the fundamental group of the unitary tangent bundle X_1 of X, let us denote by $\langle t\rangle$ the fundamental group of the fiber PSO(2) which identifies with the center of $\widetilde{G}$. The generator $t=\widetilde{k}_\pi$ identifies with a loop in X_1 corresponding to a complete clockwise rotation of unit vectors around the base point in X. It is proved in [10, Section 2] that one gets the presentation

$$(1.1)\quad \pi_1(X_1)=\langle a_1,b_1,\dots,a_g,b_g,c_1,\dots c_s,t\mid [t,a_i]=1,[t,b_i]=1,[t,c_j]=1,\ \prod_i[a_i,b_i]\prod_j c_j=t^{2g-2+s},\ c_j^{\nu_j}=t\rangle.$$

Now, let $\rho:\pi_1(X_1)\to \mathrm{GL}(V_\rho)$ be a finite-dimensional complex representation of $\pi_1(X_1)$. Since $t\in\pi_1(X_1)$ is central, $\rho(t)$ commutes with $\rho(\gamma)$ for all $\gamma\in\pi_1(X_1)$. If ρ is irreducible, $\rho(t)$ must be a scalar multiple of the identity by Schur's Lemma, say $\rho(t)=\lambda I_n$. We show that λ is in fact a root of unity. For this, let $N=\mathrm{lcm}(1,\nu_1,\dots,\nu_s)$ be the least common multiple of 1 and the orders of the elliptic conjugacy classes. The orbifold Euler characteristic $\chi(X)$ of X is given by

$$\chi(X)=2-2g+\sum_{j=1}^s\left(\frac{1}{\nu_j}-1\right)\in\mathbb{Q}.$$

Lemma 1.2.1. *Let $\rho\colon\pi_1(X_1)\to\mathrm{GL}(V)$ be an irreducible representation of dimension n. Then, $\rho(t)=\lambda I_n$ with $\lambda^{Nn\chi(X)}=1$*

Proof. Since $c_j^{\nu_j}=t$ we find $\rho(c_j)^{\nu_j}=\lambda I_n$, in particular

$$\prod_j\rho(c_j)^N=\prod_{j=1}^s\lambda^{N/\nu_j}I_n.$$

On the other hand,

$$\prod_{j=1}^s\rho(c_j)=\lambda^{2g-2+s}\prod_{i=1}^g[\rho(a_i),\rho(b_i)].$$

Since $\det(\prod_{i=1}^g[\rho(a_i),\rho(b_i)])=1$, we deduce $\lambda^{Nn\left(\sum_{j=1}^s 1/\nu_j-(2g-2+s)\right)}=1$ and the lemma follows. ☐

1.3 Representation theory of $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$

We briefly recall the principal series and the (relative) discrete series of the universal covering group of $\mathrm{PSL}(2,\mathbb{R})$, following [13, Section 1].

Let $\widetilde{M}=\widetilde{Z}$ denote the center of $\widetilde{G}$, then $\widetilde{N}\widetilde{A}\widetilde{M}$ is a parabolic subgroup of $\widetilde{G}$. The unitary dual of $\widetilde{K}\simeq\mathbb{R}$ is comprised of the unitary characters τ_m, $m\in\mathbb{R}$, defined by

$$(1.2)\qquad \tau_m(\widetilde{k}_\theta)=e^{im\theta}\qquad(\theta\in\mathbb{R}).$$

The restriction σ_ε of τ_m to $\widetilde{M}$ only depends on $\varepsilon=m+2\mathbb{Z}\in\mathbb{R}/2\mathbb{Z}$, and the unitary dual of $\widetilde{M}$ is given by all σ_ε, $\varepsilon\in\mathbb{R}/2\mathbb{Z}$.

For $\sigma = \widetilde{\sigma_\varepsilon}$, $\varepsilon \in \mathbb{R}/2\mathbb{Z}$, and $s \in \mathbb{C}$ we form the principal series representation $\pi_{\sigma,s}$ of $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$ on

$$I_{\sigma,s} = \{f \in C^\infty(\widetilde{G}) : f(na_t mg) = \sigma(m)e^{st}f(g)\},$$

acting by right translation. The $\widetilde{K}$-types in $I_{\sigma,s}$ are spanned by the functions ϕ_m with $m + 2\mathbb{Z} = \varepsilon$, where

$$\phi_m(na_t k) = e^{st}\tau_m(k_\theta) \qquad (n \in \widetilde{N}, t \in \mathbb{R}, k \in \widetilde{K}).$$

Note that the Casimir element $\Omega = \frac{1}{4}(H^2 + 2EF + 2FE)$ with

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

acts in $\pi_{\sigma,s}$ by $s(s-1)\,\mathrm{Id}$.

Unitary principal series. For $s = \frac{1}{2} + i\lambda$, $\lambda \in \mathbb{R}$, the representation $\pi_{\sigma,s}$ extends to a unitary representation on the Hilbert space

$$\left\{ f : \widetilde{G} \to \mathbb{C} : f(na_t mg) = \sigma(m)e^{(i\lambda+\frac{1}{2})t}f(g), \int_{\widetilde{M}\backslash\widetilde{K}} |f(k)|^2\, dk < \infty \right\},$$

the *unitary principal series*. We denote the distribution character of this representation by $\Theta_{\sigma,\lambda}$.

(Relative) discrete series. For $m > 0$, the $\widetilde{K}$-types $\phi_{m'}$, $m' \in \pm\{m, m+2, m+4, \ldots\}$, span an $\mathfrak{sl}(2,\mathbb{C})$-invariant subspace of $I_{\sigma_\varepsilon,\frac{m}{2}}$ where $\varepsilon = \pm m + 2\mathbb{Z} \in \mathbb{R}/2\mathbb{Z}$, which can be completed to a Hilbert space of a unitary representation of $\widetilde{G}$, the *(relative) discrete series*. We write $\Theta_{\pm m}$ for the distribution character of this representation. Details about the invariant inner product on the (relative) discrete series can be found in [13, Section 1].

2 Reidemeister torsion

In this section, we compute the Reidemeister–Turaev torsion of the unit tangent bundle X_1 of a hyperbolic orbisurface X for the Euler structure given by the geodesic flow. In Section 2.1, we define the Reidemeister–Turaev torsion. In Section 2.2, we shortly describe the two equivalent notions of Euler structures introduced by Turaev. We compute explicitly the Reidemeister-Turaev torsion in the Euler structure given by the geodesic flow in Section 2.3.

2.1 Twisted homology, Reidemeister–Turaev torsion and Euler structures

In this section, we define the twisted homology and Reidemeister torsion of a finite CW-complex W with a representation $\rho\colon \pi_1(W) \to \mathrm{GL}(V_\rho)$ and an Euler structure $\mathfrak{e}$. References include [21, 26, 27].

We will denote the fundamental group $\pi_1(W)$ by π. Let $\widetilde{W}$ be the universal cover of W. It has the structure of an (infinite) CW-complex, whose cells can be listed as follows: if $\{c^i_1, \ldots, c^i_{k_i}\}$ are the i-dimensional cells of W, then $\{\pi \cdot \widetilde{c}^i_1, \ldots, \pi \cdot \widetilde{c}^i_{k_i}\}$ is the list of the i-dimensional cells of $\widetilde{W}$, where $\pi \cdot \widetilde{c}^i_j$ denotes the set $\{\gamma \cdot \widetilde{c}^i_j \mid \gamma \in \pi\}$, see an example in Fig. 1.

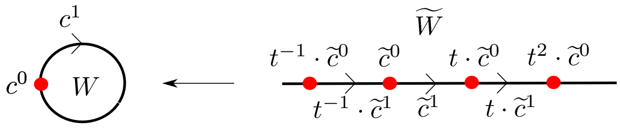

Figure 1: In this example, the CW complex W is a circle, with one 0-cell c^0 and one 1-cell c^1. Its universal cover $\widetilde{W}$ is the real line, with the action of the fundamental group $\pi_1(W) = \langle t \rangle$.

It endows $C_*(\widetilde{W})$ with the structure of a complex of $\mathbb{Z}[\pi]$-modules of finite rank, with a basis given by a choice of a lift $\widetilde{c}^i_j$ for each cell $c^i_j \in C_*(W)$. Define the complex

$$C_*(W, V_\rho) = V_\rho \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{W}).$$

Given $\{v_1, \ldots, v_n\}$ a basis of V_ρ, the set $\{v_1 \otimes \widetilde{c}^i_1, v_2 \otimes \widetilde{c}^i_1 \ldots, v_{n-1} \otimes \widetilde{c}^i_{k_i}, v_n \otimes \widetilde{c}^i_{k_i}\}$ gives a basis of the vector space $c_j(W, V_\rho)$ for each i, and the boundary operator is given by $\mathrm{id} \otimes \partial$.

Example 2.1.1. In the example of Fig. 1, given a representation $\rho\colon \pi_1(\mathbb{S}^1) \to \mathrm{GL}(V_\rho)$, the complex is just

$$C_1(\mathbb{S}^1, V_\rho) \simeq V_\rho \to V_\rho \simeq C_0(\mathbb{S}^1, V_\rho)$$

and the boundary map is $\partial(v \otimes \widetilde{c}^1) = v \otimes (t-1) \cdot \widetilde{c}^0 = (\rho(t) - I_n)v \otimes \widetilde{c}^0$. In the case $\det(\rho(t) - I_n) \neq 0$, the boundary map $\partial\colon C_1(\mathbb{S}^1, V_\rho) \to C_0(\mathbb{S}^1, V_\rho)$ is an isomorphism. The twisted homology vector spaces $H_*(\mathbb{S}^1, V_\rho)$ are then trivial, and the complex $C_*(\mathbb{S}^1, V_\rho)$ is said to be *acyclic*.

Remark 2.1.2. In Example 2.1.1, the identification $C_j(\mathbb{S}^1, V_\rho) \simeq V_\rho$ is not canonical, it depends on the choice of a lift of the cell c^i. Different choices would modify the boundary map ∂ by a factor $\rho(t^m)$, for some $m \in \mathbb{Z}$. On the other hand, the acyclicity property does of course not depend on the choices.

In general, given an acyclic based complex $(C_*, \mathbf{c}^*)$ of vector spaces over a field $\mathbb{K}$, where for each i, the set $\mathbf{c^i}$ is a basis of the vector space C_j, then the Reidemeister torsion is defined as follows. The boundary map yields exact sequences

$$0 \to Z_i \to C_i \xrightarrow{\partial_i} B_{i-1} \to 0.$$

Because the complex is acyclic, one has $B_i = Z_i$ for each i. If one picks arbitrary bases $\mathbf{b^i}$ of the spaces B_i and arbitrary lifts $\overline{\mathbf{b}}^{\mathbf{i}}$ in C_i, one obtains new bases $\mathbf{b^i} \sqcup \overline{\mathbf{b}}^{\mathbf{i-1}}$ of C_i for each i. We denote by $[\mathbf{b^i} \sqcup \overline{\mathbf{b}}^{\mathbf{i-1}} : \mathbf{c^i}]$ the determinant of the change of basis matrix, the Reidemeister torsion is defined by the alternating product

$$\mathrm{tor}(C_*, \mathbf{c}^*) = \prod_i [\mathbf{b^i} \sqcup \overline{\mathbf{b}}^{\mathbf{i-1}} : \mathbf{c^i}]^{(-1)^i} \in \mathbb{K}^*/\{\pm 1\}.$$

It is defined up to sign, it does depend on the basis $\mathbf{c}^*$, but not on the choices $\mathbf{b}^*$ nor on the lifts $\overline{\mathbf{b}}^{\mathbf{i}}$.

In the case of a CW-complex W with a representation $\rho\colon \pi_1(W) \to \mathrm{GL}(V_\rho)$, if the complex $C_*(W, V_\rho)$ is acyclic, one can compute the Reidemeister torsion after making a choice of lift for each cell of W in $\widetilde{W}$. If we denote such a choice by $\mathfrak{e}$ (for further details about the definition of the Euler structure $\mathfrak{e}$, see Section 2.2), then the torsion is denoted by $\mathrm{tor}(W, \rho, \mathfrak{e})$. Note that the torsion does not depend on the choice of basis of V_ρ.

Example 2.1.3. For the circle, with the choice of lifts $\widetilde{c}^0$ and $\widetilde{c}^1$ we made in Fig. 1, the torsion equals $\pm\frac{1}{\det(\rho(t)-I_n)}$. But if one moves the lift $\widetilde{c}^1$ to $t^m \cdot \widetilde{c}^1$, then the torsion will be multiplied by $\det \rho(t^m)$. Nevertheless, what we computed is a natural choice of torsion: one chooses *any* lift $\widetilde{c}^0$ of the base point, and then take $\widetilde{c}^1$ as on Fig. 1, taking the edge that goes out of $\widetilde{c}^0$ following the orientation. We denote this Euler structure on the circle by $\mathfrak{e}_\circ$.

2.2 Combinatorial and smooth Euler structures

In [26], Turaev investigated the concept of Euler structure. The main achievement is to show that Euler structures can be equivalently defined in two different ways. Originally Turaev's description uses simplicial structures, we slightly reformulate it in the setting of CW complexes.

An Euler chain in a CW complex W with vanishing Euler characteristic is a choice of a path c in the 1-skeleton $W'_{(1)}$ of the barycentric sub-complex W' of W, whose boundary is

$$\partial c = \sum_{\sigma \in W} (-1)^{\dim \sigma}[\sigma] \in W'_{(0)}.$$

Two such chains c and c' are equivalent, provided their difference $c - c'$ is a boundary. An equivalence class $\mathfrak{e}$ of an Euler chain is a *combinatorial* Euler structure.

Remark 2.2.1. If one fixes a 0-cell x in W, an Euler structure defines a path from x to any cell σ in W. In turn, it defines a lift $\widetilde{\sigma}$ of each cell σ to the universal cover $\widetilde{W}$ of W, and allows to define unambiguously the Reidemeister–Turaev torsion as in Section 2.1. Note that two homologous Euler structures will not necessarily define the same lifts in $\widetilde{W}$, nevertheless the value of the torsion will not be affected, since it is multiplied by the determinant of an element of the commutator subgroup $\ker(\pi_1(W) \to H_1(W))$.

On the other hand, a *smooth* Euler structure is a homology class of nowhere vanishing vector fields. Two nowhere vanishing vector fields $\mathfrak{X}$ and $\mathfrak{X}'$ on a 3-manifold X are homologous if there exists a ball $D \subset X$ such that $\mathfrak{X}$ and $\mathfrak{X}'$ are homotopic in $X \setminus D$ as nowhere vanishing vector fields.

The obstruction for two vector fields $\mathfrak{X}$ and $\mathfrak{X}'$ of being homologous is the Chern–Simons class $cs(\mathfrak{X}, \mathfrak{X}') \in H_1(X)$. It can be defined a follows: denote by $p\colon X \times [0,1] \to X$ the projection, and choose $\mathbb{X}$ a smooth section of the bundle $p^*TX \to X \times [0,1]$ which provides a homotopy between $\mathfrak{X} = \mathbb{X}|_{X\times\{0\}}$ and $\mathfrak{X}' = \mathbb{X}|_{X\times\{1\}}$ and which is transverse to the zero section. The intersection of $\mathbb{X}$ with the zero section projects onto a one-dimensional submanifold in X, whose homology class is $cs(\mathfrak{X}, \mathfrak{X}')$. It can be shown that it only depends on the homology classes of $\mathfrak{X}$ and $\mathfrak{X}'$.

It turns out that both sets of combinatorial and smooth Euler structures are affine sets on $H_1(X)$, and Turaev shows in [26, Section 6] that there is an $H_1(X)$-equivariant isomorphism from one to the other.

2.3 The torsion of the unit tangent bundle

Let X be a surface of genus g with s orbifold singularities, such that $2g-2+s > 0$. Let X_1 be the unit tangent bundle of the orbisurface X. We denote by $c_1, \dots c_s \in \pi_1(X_1)$ the elliptic elements given by loops in X around the singularities.

The aim of this section is to prove the following theorem.

Theorem 2.3.1. *Let $\rho\colon \pi_1(X_1) \to \mathrm{GL}(V_\rho)$ be an irreducible representation of dimension n, with $\rho(t) \neq I_n$. Then ρ is acyclic. For any hyperbolic metric on X, the geodesic flow on X_1 induces an Euler structure $\mathfrak{e}_{geod}$ for which*

$$\mathrm{tor}(X_1, V_\rho, \mathfrak{e}_{geod}) = \pm \frac{\det(\rho(t) - I_n)^{2g+s-2}}{\prod_{j=1}^{s} \det(\rho(c_j) - I_n)}.$$

The proof of this theorem will occupy the rest of this section. It is subdivided into two parts: first in Section 2.3.1, we decompose the unit tangent bundle X_1 into simpler pieces and exhibit a cell decomposition of each piece together with a natural choice of lifts of the cells in the universal cover. We prove Theorem 2.3.1 for this choice. In the second part Section 2.3.2, we show that this natural choice of lifts corresponds to the Euler structure induced by the geodesic flow induced by any hyperbolic metric.

2.3.1 Computation of the torsion

The 3-manifold X_1 can be cut along tori into pieces

$$X_1 = X_0 \cup N_0 \cup N_1 \cup \ldots \cup N_s, \tag{2.1}$$

where

- $X_0 \simeq (\overline{X} \setminus D_0) \times S^1$ is the product of a surface with $s+1$ boundary components with a circle;
- N_0 is a solid torus $D_0 \times S^1$;
- N_i are solid tori containing the exceptional fibers which correspond to the elliptic elements c_j.

We will use the following cell decomposition of X_0: the surface $\overline{X} \setminus D_0$ retracts on a bouquet of $2g+s$ circles, each of which can be identified with one of $a_1, \dots, b_g, c_1, \dots, c_s$. We denote the circles by $A_1, B_1, \dots A_g, B_g, C_1, \dots, C_s$, each corresponding to the element of the fundamental group of $\overline{X} \setminus D_0$ denoted by the same tiny letter. Now the product cell structure on X_0 is given by

- One 0-cell v;
- $2g+s+1$ one-cells $A_1, B_1, \dots A_g, B_g, C_1, \dots, C_s$ and T (corresponding to the circle factor);
- $2g+s$ two-cells $E_{A_1}, \dots, E_{B_g}, E_{C_1}, \dots E_{C_s}$, such that $\partial E_\gamma = \gamma T \gamma^{-1} T^{-1}$ for any $\gamma \in \{A_1, \dots, C_s\}$.

Natural lift of the cell structure. We pick an arbitrary basis $\{e_1, \dots, e_n\}$ of V_ρ, then the complex $C_*(X_0, V_\rho)$ can be computed explicitly, as in [14, Section 4], but we need to fix lifts of the cells of X_0 in the universal cover $\widetilde{X}_0$. Since X_0 has the structure of a bouquet of circles times a circle, its universal cover has the structure of a family of lattices, a portion of one of them is drawn in Fig. 2.

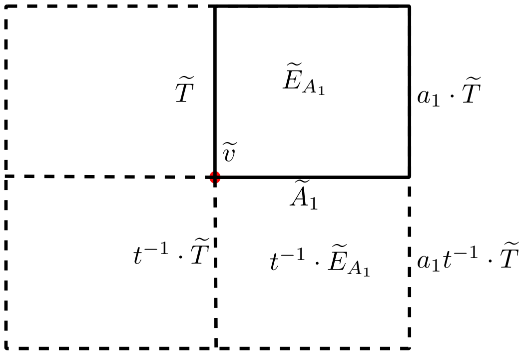

Figure 2: A sample of the universal cover of X_0. Here we draw part of a lattice corresponding to the universal cover of the two-cell E_{A_1}. The Euler structure $\mathfrak{e}_0$ is defined by taking the lifts $\widetilde{v}, \widetilde{T}, \widetilde{A}_1$ and $\widetilde{E}_{A_1}$ for each such two-dimensional cell.

We start with an arbitrary lift $\widetilde{v}$ of the unique one-cell $v \in C_0(X_0)$. Then we choose the lifts $\widetilde{A}_1, \widetilde{T}$ and $\widetilde{E}_{A_1}$ as in Fig. 2. Similarly, we take the lifts of $B_1, A_2, \dots, C_s$ and of the corresponding two-cells as above.

The set $\{e_i \otimes \widetilde{E}_\gamma \mid i = 1, \dots n,\ \gamma = A_1, \dots C_s\}$ is now a basis of $C_2(X_1, V_\rho)$. Similarly, $\{e_i \otimes \gamma \mid i = 1, \dots n,\ \gamma = \widetilde{A}_1, \dots \widetilde{C}_s, \widetilde{T}\}$ is a basis of $C_1(X_1, V_\rho)$ and $\{e_i \otimes \widetilde{v} \mid i = 1, \dots n\}$ is a basis of $C_0(X_1, V_\rho)$.

The boundary operator $\partial_2 \colon C_2(X_1, V_\rho) \to C_1(X_1, V_\rho)$ reads

$$\partial_2(e_i \otimes E_{A_1}) = e_i \otimes \left((1-t) \cdot \widetilde{A}_1 + (a_1 - 1) \cdot \widetilde{T}\right),$$

and similarly for other two-cells. Further, $\partial_1(e_i \otimes \widetilde{A}_1) = e_i \otimes (a_1 - 1)\widetilde{v}$ and similarly for the other one-cells. Hence, in the given bases, the boundary maps are given by the following matrices

$$\partial_2 = \begin{pmatrix} I_n - \rho(t) & 0 & \dots & \dots & 0 \\ 0 & I_n - \rho(t) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & I_n - \rho(t) \\ \rho(a_1) - I_n & \dots & \dots & \dots & \rho(c_s) - I_n \end{pmatrix},$$

$$\partial_1 = \begin{pmatrix} \rho(a_1) - I_n & \dots & \dots & \dots & \rho(c_s) - I_n & \rho(t) - I_n \end{pmatrix}.$$

We start with the following lemma:

Lemma 2.3.2. *Let $\rho \colon \pi_1(X_1) \to \mathrm{GL}(V_\rho)$ be an n-dimensional irreducible representation. If $\det(\rho(t) - I_n) \neq 0$, then the complex $C_*(X_1, V_\rho)$ is acyclic.*

Proof. For $i = 0, \dots, s$, the solid torus N_i has the homotopy type of a circle, hence the complex $C_*(N_i, V_\rho)$ is just

$$C_1(N_i, V_\rho) \simeq V_\rho \xrightarrow{\rho(c_i)-I_n} V_\rho \simeq C_0(N_i, V_\rho)$$

where $\rho(c_0)$ is understood as being $\rho(t)$. Since $\rho(c_i)^{\nu_i} = \rho(t)$, the assumption $\det(\rho(t) - I_n) \neq 0$ implies $\det(\rho(c_i) - I_n) \neq 0$ for any i. Hence, we showed that $C_*(N_i, V_\rho)$ is acyclic when $\det(\rho(t) - I_n) \neq 0$. With the same argument, one shows that $C_*(\partial N_i, V_\rho)$ is acyclic for any i.

Now we prove that $C_*(X_0, V_\rho)$ is acyclic. First, the map ∂_1 is surjective since $\det(\rho(t) - I_n) \neq 0$, hence $H_0(X_0, V_\rho) = \{0\}$. Moreover, the $n(2g+s) \times n(2g+s)$ diagonal submatrix

$$\partial_2' = \begin{pmatrix} I_n - \rho(t) & 0 & \dots & \dots & 0 \\ 0 & I_n - \rho(t) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & I_n - \rho(t) \end{pmatrix}$$

of ∂_2 has determinant equal to $\det(I_n - \rho(t))^{2g+s}$, in particular it is non-zero and ∂_2 is injective, i.e., $H_2(X_0, V_\rho) = \{0\}$. Since the Euler characteristic of X_0 vanishes, we conclude that $H_1(X_0, V_\rho) = \{0\}$ and the acyclicity follows. By a Mayer–Vietoris argument one deduces that $C_*(X_1, V_\rho)$ is acyclic. □

Proof of Theorem 2.3.1. First, because $\rho(t) \neq I_n$ is central, $\det(I_n - \rho(t)) \neq 0$. Now we use again a Mayer–Vietoris argument. For any solid torus N_i in the decomposition given in Eq. (2.1), we compute the torsion as in Example 2.1.3 in the natural Euler structure $\mathfrak{e}_\circ$. Denote by ℓ_i the core of the solid torus N_i. Since $c_i^{\nu_i} t^{-1}$ is a meridian of N_i, the curve ℓ_i is given by $\ell_i = c_i^{n_i} t^{m_i}$ with $m_i \nu_i + n_i = 1$. Writing $t = c_i^{\nu_i}$, one sees that $\ell_i = c_i$ is the core of the solid torus N_i. For the torsion we obtain

$$\mathrm{tor}(N_i, V_\rho, \mathfrak{e}_\circ) = \pm \frac{1}{\det(I_n - \rho(c_i))},$$

where c_0 is understood as being t. On the other hand, using [27, Theorem 2.2], one computes the torsion of X_0 in the Euler structure $\mathfrak{e}_0$:

$$\mathrm{tor}(X_0, V_\rho, \mathfrak{e}_0) = \pm \frac{\det(\partial_2')}{\det(\rho(t) - I_n)}$$

and it follows

$$\mathrm{tor}(X_0, V_\rho, \mathfrak{e}_0) = \pm \det(\rho(t) - I_n)^{2g+s-1}.$$

Finally, we use the well-known multiplicativity of the torsion ([27, Theorem 1.5]). We denote by $\mathfrak{e}$ the natural Euler structure on X induced by $\mathfrak{e}_0$ on X_0 (which induces an Euler structure on each boundary torus ∂N_i compatible with $\mathfrak{e}_\circ$). We have:

$$\mathrm{tor}(X_1, V_\rho, \mathfrak{e}) = \pm \frac{\mathrm{tor}(X_0, V_\rho, \mathfrak{e}_0) \prod_{i=0}^{s} \mathrm{tor}(N_i, V_\rho, \mathfrak{e}_\circ)}{\prod_{i=0}^{s} \mathrm{tor}(\partial N_i, V_\rho, \mathfrak{e})}.$$

For the boundary tori ∂N_i an explicit computation yields $\mathrm{tor}(\partial N_i, V_\rho, \mathfrak{e}) = 1$. We deduce

$$\mathrm{tor}(X_1, V_\rho, \mathfrak{e}) = \pm \frac{\det(\rho(t) - I_n)^{2g+s-2}}{\prod_{j=1}^{s} \det(\rho(c_j) - I_n)}.$$ □

2.3.2 Computation of the Euler structure

Now we prove that the Euler structure $\mathfrak{e}$ corresponds to the geodesic flow through Turaev's isomorphism between combinatorial and smooth Euler structure. To do so, we first exhibit an Euler chain realizing $\mathfrak{e}$, and construct a non-vanishing vector field $\mathfrak{X}_{\mathfrak{e}}$ realizing the corresponding smooth Euler structure, following the construction of Turaev in [26, Section 6]. Then, we compute the Chern–Simons class $cs(\mathfrak{X}_{geod}, \mathfrak{X}_{\mathfrak{e}})$ giving the obstruction for the geodesic flow $\mathfrak{X}_{geod}$ being homologous to the vector field $\mathfrak{X}_{\mathfrak{e}}$ and show it is trivial.

The Euler chain. The manifold X_0 retracts on a CW-complex given by a gluing of tori along a circle, on each the Euler chain is described as follows (see Fig. 3): we fix a basepoint v at the intersection of each torus and make a straight path to the barycenter of each cell, oriented negatively for the one-cells and positively for the two-cell. The result is a path c with boundary $\partial c = [v] - [T] - [\gamma] + [E_\gamma]$, hence an Euler chain. It is clear that it induces the Euler structure $\mathfrak{e}_0$ on X_0.

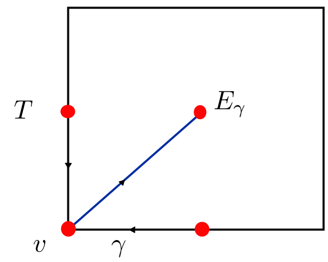

Figure 3: For any one-cell γ in $\{A_1, \dots, C_s\}$, we draw the corresponding two-cell E_γ with an Euler chain joining the vertex v with each cell, oriented accordingly to the dimensions as in Section 2.2.

On the solid tori we just take the natural Euler chain denoted $\mathfrak{e}_\circ$ in Example 2.1.3.

A vector field realizing $\mathfrak{e}$. We construct a non-vanishing vector field on the CW-complex for X_0. On each two-cell as in Fig. 3, we start with a Stiefel vector field: the barycenter of each cell σ is a zero of Hopf index $(-1)^{\dim \sigma}$, and the vector field flows from the lower dimensional cells to the higher dimensional cells (see [21, Section 3.2]). Then we modify it as depicted in [21, Figure 3.5]: around each barycenter we modify it so that the index in the cone neighborhood of an edge joining two barycenters is the coefficient of this one-cell in the Euler chain c. In particular it remains untouched except for the 3 one-cells involved in Fig. 3.

It follows from the fact that c is an Euler chain that the index of the new vector field is zero around each vertex, hence the new vector field extends to a non-vanishing vector field $\mathfrak{X}_{\mathfrak{e}_0}$ on the whole 2-dimensional CW complex whose homology class does not depend on the choices we made.

On each solid torus N_i in the decomposition given in Eq. (2.1), the natural Euler structure $\mathfrak{e}_\circ$ yields a vector field $\mathfrak{X}_\circ$ tangent to the core of the solid torus. We extend it to the whole solid torus by a radial homotopy $r\mathfrak{X}_{e_0} + (1-r)\mathfrak{X}_\circ$, on any disc section of the torus. The vector field we obtain equals $\mathfrak{X}_{\mathfrak{e}_0}$ on the boundary.

Gluing all together, it defines an Euler structure $\mathfrak{e}$ on the manifold X_1.

The Chern-Simons class $cs(\mathfrak{X}_{geod}, \mathfrak{X}_{\mathfrak{e}})$. Now we prove that the vector field we computed is homologous to the geodesic flow $\mathfrak{X}_{geod}$. We denote by $\mathfrak{X}_\theta$ the vector field directed everywhere positively along the fiber of the unit bundle X_1. Observe that the geodesic flow $\mathfrak{X}_{geod}$ and $\mathfrak{X}_\theta$ are homotopic as nowhere vanishing vector fields: it is clear on each solid torus N_i, and a nowhere vanishing homotopy can be realized as $t\mathfrak{X}_\theta + (1-t)\mathfrak{X}_{geod}$ on X_0.

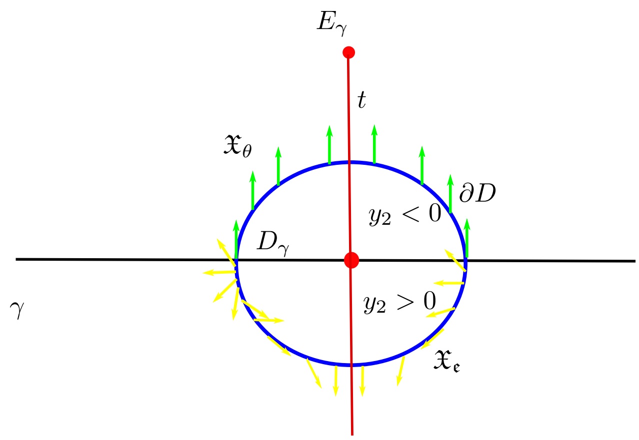

Figure 4: We draw the vector fields $\mathfrak{X}_\theta$ and $\mathfrak{X}_{\mathfrak{e}}$ in the 2-cell E_γ for $\gamma \in \{a_1, \ldots, a_g, b_1, \ldots, b_g, c_1, \ldots, c_s\}$. More precisely, we describe them on a spherical neighborhood of the vertex corresponding to γ. The red segment represents the homology class of t. After a homotopy of the yellow vector field $\mathfrak{X}_{\mathfrak{e}}$, it coincides with the green vector field $\mathfrak{X}_\theta$ on the two extremal points ∂D_γ at the intersection between the black segment γ and the blue circle ∂D. The new vector field obtained by gluing these two has index 0 on the circle ∂D, hence the homology class $[t]$ does not contribute to $cs(\mathfrak{X}_\theta, \mathfrak{X}_{\mathfrak{e}})$.

Hence, we are led to compute $cs(\mathfrak{X}_{\mathfrak{e}}, \mathfrak{X}_\theta)$. We follow the procedure given by Turaev in [26, Section 5.2]. We compute the Chern–Simons class on each two-cell. In Fig. 4 we calculate the contribution of the homology class of t in one of the 2-cells, and show it is trivial. The same computation for the homology class of any γ shows

$$cs(\mathfrak{X}_\theta, \mathfrak{X}_{\mathfrak{e}}) = 0$$

hence

$$cs(\mathfrak{X}_{geod}, \mathfrak{X}_{\mathfrak{e}}) = 0.$$

It concludes the proof of Theorem 2.3.1, since it shows that the Euler structure $\mathfrak{e}$ used in Section 2.3.1 is induced by the geodesic flow.

Remark 2.3.3. Unlike the manifold X_1, the pieces X_0 and N_i are manifolds with boundary. In this case, Turaev introduces a thickening of the boundary so that the vector fields representing Euler structure are transverse to the boundary.

Here, we deal with vector fields tangent everywhere to the boundary. It makes the description much simpler, and we can recover the setting of Turaev by taking a thickening of the boundaries of X_0 and of each N_0 and by extending the vector fields so that

- Each vector field is tangent to the slice $\partial\Theta \times \{0\}$ for $\Theta \in \{X_0, N_0, \dots, N_s\}$.
- The vector field is pointing outward on $\partial X_0 \times \{1\}$.
- The vector field are pointing inward on $\partial N_i \times \{1\}, i = 0, \dots s$.

3 A Selberg trace formula for non-unitary twists

We introduce the twisted Laplacian on sections of vector bundles over X and use it to derive a trace formula for the corresponding heat operator. In Section 3.1, we define the vector bundles we will make use of, and in Section 3.2, the corresponding twisted Laplace operators. Then, in Section 3.3, we write the pre-trace formula for the corresponding heat operator, and we compute the contribution of the identity (Section 3.4), the hyperbolic (Section 3.5) and elliptic (Section 3.6) contributions. We gather all this in Section 3.7 to write down the Selberg trace formula for (a shift of) the twisted Laplacian. Finally, in Section 3.8, we prove a technical result on multiplicities of the eigenvalues of the Laplacians that will be used in Section 4 to prove the meromorphicity of the twisted Ruelle zeta function.

3.1 Vector bundles over X

Consider a finite-dimensional, complex representation $\rho : \widetilde{\Gamma} \to \mathrm{GL}(V_\rho)$. Let

$$E_\rho = V_\rho \times_{\widetilde{\Gamma}} \widetilde{G} \to \widetilde{\Gamma}\backslash\widetilde{G} = X_1$$

denote the associated flat vector bundle over X_1 and equip it with a flat connection ∇^{E_χ}. In general, this vector bundle does not define a vector bundle over $X = X_1/\widetilde{K}$, because both $\widetilde{\Gamma}$ and $\widetilde{K}$ contain the center $\widetilde{Z}$ of $\widetilde{G}$ and ρ is not necessarily trivial on $\widetilde{Z}$. In order to obtain a vector bundle on X, we have to twist this construction by a character of $\widetilde{K}$ which is compatible with ρ on $\widetilde{Z}$. We therefore assume that $\rho(t) = e^{-im\pi}$ for some $m \in \mathbb{R}$. Note that this is automatically satisfied if ρ is irreducible by Lemma 1.2.1.

Let $\tau = \tau_m : \widetilde{K} \to \mathrm{U}(V_\tau)$ denote the character of $\widetilde{K}$ defined in Eq. (1.2), then $\tau(t) = \rho(t)^{-1}$. We consider the homogeneous vector bundle

$$E_\tau = \widetilde{G} \times_{\widetilde{K}} V_\tau \to \mathbb{H} = \widetilde{G}/\widetilde{K}.$$

The invariant inner product on V_τ induces a Hermitian metric and a metric connection ∇^{E_τ} on E_τ. By the same reason as above, the bundle E_τ does not factor through $X = \widetilde{\Gamma}\backslash\widetilde{G}/\widetilde{K}$.

In order to obtain a vector bundle on X associated with ρ, we consider the tensor product representation $V_\tau \otimes V_\rho$ of $\widetilde{K} \times \widetilde{\Gamma}$ and let

$$E_{\tau,\rho} = \widetilde{G} \times_{\widetilde{K}\times\widetilde{\Gamma}} (V_\tau \otimes V_\rho) \to \widetilde{\Gamma}\backslash\widetilde{G}/\widetilde{K},$$

where $\widetilde{K} \times \widetilde{\Gamma}$ acts on $\widetilde{G}$ by $g \cdot (k, \gamma) = \gamma^{-1} g k$. Note that this action is not effective, but the elements in $\widetilde{K} \times \widetilde{\Gamma}$ acting trivially on $\widetilde{G}$ are of the form (z, z^{-1}), $z \in \widetilde{Z}$, and hence also act trivially on $V_\tau \otimes V_\rho$ by construction. Smooth sections of the bundle $E_{\tau,\rho}$ can be identified with smooth functions $f : \widetilde{G} \to V_\tau \otimes V_\rho$, such that

$$f(\gamma g k) = \left[\tau(k)^{-1} \otimes \rho(\gamma)\right] f(g) \qquad (g \in \widetilde{G}, k \in \widetilde{K}, \gamma \in \widetilde{\Gamma}).$$

We now define a connection $\nabla^{E_{\tau,\rho}}$ on $E_{\tau,\rho}$ in terms of its associated covariant derivative. For this we identify vector fields $\mathfrak{X} \in C^\infty(X, TX)$ on X with smooth functions $\mathfrak{X} : \widetilde{G} \to \mathfrak{p}$ such that $\mathfrak{X}(\gamma g k) = \mathrm{Ad}(k)|_{\mathfrak{p}}^{-1} \mathfrak{X}(g)$. Then,

$$\nabla^{E_{\tau,\rho}}_{\mathfrak{X}} f(g) = \left.\frac{d}{dt}\right|_{t=0} f(g \exp t\mathfrak{X}(g)) \qquad (f \in C^\infty(X, E_{\tau,\rho}), \mathfrak{X} \in C^\infty(X, TX))$$

defines a covariant derivative and hence a connection on $E_{\tau,\rho}$. Note that in general there is no Hermitian metric on $E_{\tau,\rho}$ compatible with the connection, since the representation ρ is not necessarily unitary.

3.2 The twisted Laplacian

For a complex vector bundle $E \to X$ with covariant derivative ∇^E, the second covariant derivative $(\nabla^E)^2$ is defined by

$$(\nabla^E)^2_{\mathfrak{X},\mathfrak{X}'} = \nabla^E_{\mathfrak{X}} \nabla^E_{\mathfrak{X}'} - \nabla^E_{\nabla^{\mathrm{LC}}_{\mathfrak{X}} \mathfrak{X}'} \qquad (\mathfrak{X}, \mathfrak{X}' \in C^\infty(X, TX)),$$

where ∇^{LC} is the Levi–Civita connection on TX. The negative of the trace of the second covariant derivative is the corresponding connection Laplacian:

$$\Delta_E = -\operatorname{tr}\left((\nabla^E)^2\right).$$

For ρ and τ as in the previous section, the twisted Laplacian $\Delta^\sharp_{\tau,\rho}$ is defined as the connection Laplacian

$$\Delta^\sharp_{\tau,\rho} = \Delta_{E_{\tau,\rho}} = -\operatorname{tr}\left((\nabla^{E_{\tau,\rho}})^2\right).$$

Note that the orbisurface X has a finite cover, which is a manifold. Therefore, the spectral theory of $\Delta^\sharp_{\tau,\rho}$ on X is the same as the spectral theory of its lift to the finite cover, acting on sections invariants under the action of the finite group of Deck transformations. Hence, by [15, Theorem 4.3], we have the following properties (see also for the manifold case the work of Müller [17], and for the orbifold case the work of Fedosova [6] and Shen [23, Section 7]):

(1) If we choose a Hermitian metric on $E_{\tau,\rho}$, then $\Delta^\sharp_{\tau,\rho}$ acts in $L^2(X, E_{\tau,\rho})$ with domain $C^\infty(X, E_{\tau,\rho})$. However, it is not a formally self-adjoint operator in general. Note that while the inner product on $L^2(X, E_{\tau,\rho})$ depends on the chosen inner product on V_ρ, the space $L^2(X, E_{\tau,\rho})$ as a topological vector space does not, thanks to the compactness of X.

(2) $\Delta^\sharp_{\tau,\rho}$ is an elliptic second order differential operator with purely discrete spectrum $\mathrm{spec}(\Delta^\sharp_{\tau,\rho}) \subseteq \mathbb{C}$, consisting of generalized eigenvalues. The spectrum is contained in a translate of a positive cone in $\mathbb{C}$. The generalized eigenspaces

$$\{f \in L^2(X, E_{\tau,\rho}) : (\Delta^\sharp_{\tau,\rho} - \mu \, \mathrm{Id})^N f = 0 \text{ for some } N\}$$

are finite-dimensional, contained in $C^\infty(X, E_{\tau,\rho})$, and their direct sum, as μ runs through $\operatorname{spec}(\Delta^\sharp_{\tau,\rho})$, is dense in $L^2(X, E_{\tau,\rho})$.

(3) If $\widetilde{E}_{\tau,\rho} \simeq E_\tau \times V_\rho \to \mathbb{H}$ denotes the pullback of $E_{\tau,\rho}$ to $\mathbb{H}$, the lift $\widetilde{\Delta}^\sharp_{\tau,\rho}$ of $\Delta^\sharp_{\tau,\rho}$ to $\mathbb{H}$ takes the form

$$\widetilde{\Delta}^\sharp_{\tau,\rho} = \Delta_\tau \otimes \mathrm{Id}_{V_\rho}, \tag{3.1}$$

where Δ_τ denotes the Laplace–Beltrami operator on $E_\tau \to \mathbb{H}$.

3.3 The pre-trace formula

The heat operator $e^{-t\Delta^\sharp_{\tau_m,\rho}}$ corresponding to the twisted Laplacian $\Delta^\sharp_{\tau_m,\rho}$ is of trace class and by Eq. (3.1) its integral kernel is the smooth function

$$H_t^{\tau_m,\rho}(g,h) = \sum_{[\gamma]\subseteq\widetilde{\Gamma}/\widetilde{Z}} \rho(\gamma) \otimes H_t^{\tau_m}(h^{-1}\gamma g) \qquad (g,h \in \widetilde{G}),$$

where the summation is over the conjugacy classes of $\widetilde{\Gamma}/\widetilde{Z} \simeq \Gamma$ and

$$H_t^{\tau_m} \in (C^\infty(\widetilde{G}) \otimes \mathrm{End}(V_{\tau_m}))^{\widetilde{K}\times\widetilde{K}}$$

denotes the heat kernel of the Laplace–Beltrami operator on $E_{\tau_m} \to \mathbb{H}$. Note that the center $\widetilde{Z}$ of $\widetilde{G}$ lies in $\widetilde{K}$, hence each summand is independent on the choice of γ relatively to $\widetilde{Z}$. The trace of $e^{-t\Delta^\sharp_{\tau_m,\rho}}$ can be computed in two different ways, by summing over the generalized eigenvalues of $\Delta^\sharp_{\tau_m,\rho}$ (the spectral side) and by integrating the heat kernel along the diagonal in $\widetilde{\Gamma}\backslash\widetilde{G}$ (the geometric side). For the spectral side we denote for an eigenvalue $\mu \in \operatorname{spec}(\Delta^\sharp_{\tau_m,\rho})$ its algebraic multiplicity by

$$\operatorname{mult}(\mu; \Delta^\sharp_{\tau_m,\rho}) = \dim\{f \in L^2(X, E_{\tau_m,\rho}) : (\Delta^\sharp_{\tau_m,\rho} - \mu\,\mathrm{Id})^N f = 0 \text{ for some } N\}.$$

Note that $\operatorname{mult}(\mu; \Delta^\sharp_{\tau_m,\rho})$ is independent of the chosen metric on E_ρ. For the geometric side, we use the standard arguments grouping conjugacy classes in $\widetilde{\Gamma}/\widetilde{Z}$ (see e.g., [28, Section 2]). This leads to the following pre-trace formula which was made rigorous by Müller [17, Proposition 5.1] for manifolds and generalized to orbifolds by Shen [23, Theorem 7.1]:

$$\begin{aligned} &\sum_{\mu\in\operatorname{spec}(\Delta^\sharp_{\tau_m,\rho})} \operatorname{mult}(\mu; \Delta^\sharp_{\tau_m,\rho}) e^{-t\mu} \\ &\qquad = \sum_{[\gamma]\subseteq\widetilde{\Gamma}/\widetilde{Z}} \operatorname{Vol}(\widetilde{\Gamma}_\gamma\backslash\widetilde{G}_\gamma)\,\operatorname{tr}\rho(\gamma) \int_{\widetilde{G}_\gamma\backslash\widetilde{G}} \operatorname{tr} H_t^{\tau_m}(g^{-1}\gamma g)\,d\dot{g}. \end{aligned} \tag{3.2}$$

Since $\widetilde{\Gamma}/\widetilde{Z} \simeq \Gamma$ consists of the identity element e, hyperbolic and elliptic elements, we can compute the three contributions to the right-hand side in (3.2) separately.

Remark 3.3.1. Since $m \in \mathbb{Q}$ we could just as well formulate everything in terms of a finite covering group G_1 of G which is a connected, semisimple group with finite center. This formulation allows us to use the Harish-Chandra L^q-Schwartz space $\mathcal{C}^q(G_1)$ (see e.g. [1, p. 161–162] for its definition). By [1, Lemma 2.3 and Proposition 2.4], the heat kernel $H_t^{\tau_m}$ is contained in $\mathcal{C}^q(G_1) \otimes \mathrm{End}(V_{\tau_m})$ for any $q > 0$, and hence it is an admissible function in the sense of Gangolli [11, p. 407]. We can therefore apply the distribution character of a unitary representation of G_1 to $H_t^{\tau_m}$ in Section 3.4, Section 3.5 and Section 3.6.

3.4 The identity contribution

The contribution of $\gamma = e$ to the right-hand side of (3.2) is

$$\mathrm{Vol}(X)\dim(V_\rho)H_t^{\tau_m}(e).$$

We can employ the Plancherel formula for $L^2(\mathbb{H}, E_{\tau_m})$ from [13, Lemma 5] to find that

$$H_t^{\tau_m}(e) = \frac{1}{4\pi}\left[\int_{\mathbb{R}} \Theta_{\sigma,\lambda}(H_t^{\tau_m})\frac{\lambda\sinh 2\pi\lambda}{\cosh 2\pi\lambda + \cos\pi m}\,d\lambda + \sum_{n\equiv m+1 \mod 2} |n|\Theta_{n+\mathrm{sign}(n)}(H_t^{\tau_m})\right],$$

where σ is the restriction of τ_m to $\widetilde{M}$. By [20, p. 2634] we find that

$$\Theta_{\sigma,\lambda}(H_t^{\tau_m}) = e^{-t(\lambda^2+\frac{1}{4})} \tag{3.3}$$

and

$$\Theta_{m'}(H_t^{\tau_m}) = \begin{cases} e^{-\frac{|m'|}{2}(1-\frac{|m'|}{2})t} & \text{for } m = m' + 2\,\mathrm{sign}(m')\ell,\ \ell \in \mathbb{Z}_{\geq 0}, \\ 0 & \text{else.}\end{cases} \tag{3.4}$$

Note that, in contrast to the case of for instance higher-dimensional hyperbolic manifolds, the expression for $\Theta_{\sigma,\lambda}(H_t^{\tau_m})$ is independent of σ since $\widetilde{M}$ is discrete. Write $n + \mathrm{sign}(n) = \mathrm{sign}(m)(|m| - (\ell - 1))$, $1 \leq \ell < |m|$ odd, then the identity contribution becomes

$$\frac{\mathrm{Vol}(X)\dim(V_\rho)}{4\pi}\left[\int_{\mathbb{R}} e^{-t(\lambda^2+\frac{1}{4})}\frac{\lambda\sinh 2\pi\lambda}{\cosh 2\pi\lambda + \cos\pi m}\,d\lambda + \sum_{\substack{1\leq\ell<|m| \\ \ell \text{ odd}}} (|m| - \ell)e^{\frac{(|m|-\ell+1)(|m|-\ell-1)}{4}t}\right].$$

3.5 The hyperbolic contribution

For a conjugacy class $[\gamma] \subseteq \widetilde{\Gamma}/\widetilde{Z}$ of hyperbolic elements we choose a representative $\gamma \in \widetilde{\Gamma}$ which is conjugate in $\widetilde{G}$ to $\widetilde{a}_{\ell(\gamma)}$, instead of $z\widetilde{a}_{\ell(\gamma)}$, $z \in \widetilde{Z}$. Here, $\ell(\gamma) > 0$

is the length of the geodesic corresponding to $[\gamma]$. Then, by [13, Lemma 2] and [28, Section 6], the contribution of $[\gamma]$ to the right-hand side of (3.2) is given by

$$\operatorname{tr}\rho(\gamma)\frac{l(\gamma)}{n_\Gamma(\gamma)D(\gamma)}\frac{1}{2\pi}\int_{\mathbb{R}}\Theta_{\sigma,\lambda}(H_t^{\tau_m})e^{-il(\gamma)\lambda}\,d\lambda,$$

where σ is the restriction of τ_m to $\widetilde{M}$, $n_\Gamma(\gamma)$ is the multiplicity of the (not necessarily prime) geodesic $[\gamma]$ and $D(\gamma)=2\sinh\frac{l(\gamma)}{2}$ is the Weyl denominator. By (3.3), the total hyperbolic contribution becomes

$$\sum_{[\gamma]\text{ hyp.}}\operatorname{tr}\rho(\gamma)\frac{l(\gamma)}{n_\Gamma(\gamma)D(\gamma)}\frac{1}{2\pi}\int_{\mathbb{R}}e^{-t(\lambda^2+\frac{1}{4})}e^{-il(\gamma)\lambda}\,d\lambda.$$

Computing the integral, we obtain

$$\frac{e^{-\frac{t}{4}}}{2\sqrt{4\pi t}}\sum_{[\gamma]\text{ hyp.}}\frac{l(\gamma)\operatorname{tr}\rho(\gamma)}{n_\Gamma(\gamma)\sinh\frac{l(\gamma)}{2}}e^{-\frac{l(\gamma)^2}{4t}}.$$

We remark that this sum converges absolutely by the same arguments as in [8, Proposition 4.1.1].

3.6 The elliptic contribution

For a conjugacy class $[\gamma]\subseteq\widetilde{\Gamma}/\widetilde{Z}$ of elliptic elements, we choose a representative $\gamma\in\widetilde{\Gamma}$ which is conjugate in $\widetilde{G}$ to $\widetilde{k}_{\theta(\gamma)}$ with $\theta(\gamma)\in(0,\pi)$. Let $M(\gamma)$ denote the order of the stabilizer $\widetilde{\Gamma}_\gamma/\widetilde{Z}$ of γ in $\widetilde{\Gamma}/\widetilde{Z}$, then

$$\operatorname{Vol}(\widetilde{\Gamma}_\gamma\backslash\widetilde{G}_\gamma)=\operatorname{Vol}(\Gamma_\gamma\backslash G_\gamma)=\operatorname{Vol}(\langle k_{\theta(\gamma)}\rangle\backslash\operatorname{PSO}(2))=\frac{1}{M(\gamma)}$$

and hence, by [13, Lemma 4] the contribution of $[\gamma]$ to the pre-trace formula (3.2) is given by

$$\begin{aligned}\frac{\operatorname{tr}\rho(\gamma)}{M(\gamma)}\cdot\frac{1}{2i\sin\theta(\gamma)}\Bigg[\frac{i}{2}\int_{\mathbb{R}}\Theta_{\sigma,\lambda}(H_t^{\tau_m})\frac{\cosh 2(\pi-\theta(\gamma))\lambda+e^{i\pi m}\cosh 2\theta(\gamma)\lambda}{\cosh 2\pi\lambda+\cos\pi m}\,d\lambda\\ -\delta_{(|m|-1)\bmod 2}\frac{\Theta_1(H_t^{\tau_m})-\Theta_{-1}(H_t^{\tau_m})}{2}\\ -\sum_{\substack{n\equiv m+1\bmod 2\\ n\neq 0}}\operatorname{sign}(n)e^{in\theta(\gamma)}\Theta_{n+\operatorname{sign}(n)}(H_t^{\tau_m})\Bigg],\end{aligned}$$

where $\delta_{(|m|-1)\bmod 2}$ is meant to be 1 for $|m|-1\equiv 0 \mod 2$ and 0 otherwise. Using (3.3) and (3.4), the elliptic contribution becomes

$$\begin{aligned}\sum_{[\gamma]\text{ ell.}}\frac{\operatorname{tr}\rho(\gamma)}{4M(\gamma)\sin\theta(\gamma)}\Bigg[\int_{\mathbb{R}}e^{-t(\lambda^2+\frac{1}{4})}\frac{\cosh 2(\pi-\theta(\gamma))\lambda+e^{i\pi m}\cosh 2\theta(\gamma)\lambda}{\cosh 2\pi\lambda+\cos\pi m}\,d\lambda\\ +2i\operatorname{sign}(m)\sum_{\substack{1\leq\ell<|m|\\ \ell\text{ odd}}}e^{i\operatorname{sign}(m)(|m|-\ell)\theta(\gamma)}e^{\frac{(|m|-\ell+1)(|m|-\ell-1)}{4}t}\Bigg].\end{aligned}$$

Note that this sum is always finite since Γ only contains finitely many elliptic elements.

3.7 The trace formula

Adding the identity, the hyperbolic and the elliptic contribution finally shows the trace formula. For the statement let

$$A^\sharp_{\tau_m,\rho} = \Delta^\sharp_{\tau_m,\rho} - \frac{1}{4}.$$

Theorem 3.7.1 (Selberg trace formula with non-unitary twist)**.** *Let $\rho : \widetilde{\Gamma} \to \mathrm{GL}(V_\rho)$ be a finite-dimensional complex representation with $\rho(t) = e^{-im\pi}$, $m \in \mathbb{R}$. Then, for every $t > 0$ we have*

$$\begin{aligned}
&\mathrm{Tr}(e^{-tA^\sharp_{\tau_m,\rho}}) \\
&\quad = \frac{\mathrm{Vol}(X)\dim(V_\rho)}{4\pi}\Bigg[\int_{\mathbb{R}} e^{-t\lambda^2}\frac{\lambda\sinh 2\pi\lambda}{\cosh 2\pi\lambda + \cos\pi m}\,d\lambda \\
&\qquad\qquad + \sum_{\substack{1\leq\ell<|m| \\ \ell \text{ odd}}} (|m|-\ell)e^{(\frac{|m|-\ell}{2})^2 t}\Bigg]. \\
&\qquad + \frac{1}{2\sqrt{4\pi t}}\sum_{[\gamma] \text{ hyp.}} \frac{l(\gamma)\,\mathrm{tr}\,\rho(\gamma)}{n_\Gamma(\gamma)\sinh\frac{l(\gamma)}{2}} e^{-\frac{l(\gamma)^2}{4t}} \\
&\qquad + \sum_{[\gamma] \text{ ell.}} \frac{\mathrm{tr}\,\rho(\gamma)}{4M(\gamma)\sin\theta(\gamma)}\Bigg[\int_{\mathbb{R}} e^{-t\lambda^2}\frac{\cosh 2(\pi-\theta(\gamma))\lambda + e^{i\pi m}\cosh 2\theta(\gamma)\lambda}{\cosh 2\pi\lambda + \cos\pi m}\,d\lambda \\
&\qquad\qquad + 2i\,\mathrm{sign}(m)\sum_{\substack{1\leq\ell<|m| \\ \ell \text{ odd}}} e^{i\,\mathrm{sign}(m)(|m|-\ell)\theta(\gamma)}e^{(\frac{|m|-\ell}{2})^2 t}\Bigg].
\end{aligned}$$

Here, the summation is over the conjugacy classes $[\gamma]$ of hyperbolic, resp. elliptic, elements in $\widetilde{\Gamma}/\widetilde{Z} \simeq \Gamma$ and the representative $\gamma \in \widetilde{\Gamma}$ is chosen such that it is conjugate in $\widetilde{G}$ to $\widetilde{a}_{\ell(\gamma)}$ ($\ell(\gamma) > 0$), resp. $\widetilde{k}_{\theta(\gamma)}$ ($\theta(\gamma) \in (0,\pi)$).

3.8 An application to the determination of multiplicities

In order to conclude that the residues of the logarithmic derivative of the Selberg zeta function are integers, we now deduce from the trace formula an expression for the difference of the multiplicities $\mathrm{mult}(\mu;\Delta^\sharp_{\tau_m,\rho})$ for m and $m+2$. For this, it is more convenient to work with $A^\sharp_{\tau_m,\rho}$. For $\mu \in \mathbb{C}$ we let

$$L^2(X,E_{\tau_m,\rho})_\mu = \{f \in L^2(X,E_{\tau_m,\rho}) : (A^\sharp_{\tau_m,\rho} - \mu\,\mathrm{Id})^N f = 0 \text{ for some } N\}$$

denote the corresponding generalized eigenspace of $A^\sharp_{\tau_m,\rho}$ and

$$\mathrm{mult}(\mu;A^\sharp_{\tau_m,\rho}) = \dim L^2(X,E_{\tau_m,\rho})_\mu$$

its multiplicity.

Define $\kappa, x_\pm \in \mathfrak{sl}(2,\mathbb{C})$ by

$$\kappa = E - F, \qquad x_\pm = \frac{1}{2}(H \pm i(E+F)),$$

then

$$[\kappa, x_\pm] = \pm 2i x_\pm, \qquad [x_+, x_-] = -i\kappa.$$

We remark that the complexification $\mathfrak{sl}(2,\mathbb{C})$ of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of G has to be used here since $\mathrm{ad}(\kappa)$ does not act diagonalizably on $\mathfrak{sl}(2,\mathbb{R})$. The Casimir element Ω for $\mathfrak{sl}(2,\mathbb{C})$ can be expressed in terms of the two bases H, E, F and κ, x_+, x_- as

$$\Omega = \frac{1}{4}(H^2 + 2EF + 2FE) = -\frac{1}{4}(\kappa^2 - 2x_+x_- - 2x_-x_+).$$

Moreover, we can write

$$\Omega = -\frac{1}{4}\kappa^2 - \frac{1}{2}i\kappa + x_-x_+ = -\frac{1}{4}\kappa^2 + \frac{1}{2}i\kappa + x_+x_-. \tag{3.5}$$

For $u \in C^\infty(\mathbb{H}, E_{\tau_m}) = (C^\infty(G) \otimes V_{\tau_m})^K$ we define $\partial_\pm u(g) = d\pi_R(x_\pm)u(g)$, where $d\pi_R$ denotes the Lie algebra representation corresponding to the right regular representation π_R of G on $C^\infty(G)$. Then $[\kappa, x_\pm] = \pm 2ix_\pm$ implies

$$\partial_\pm : C^\infty(\mathbb{H}, E_{\tau_m}) \to C^\infty(\mathbb{H}, E_{\tau_{m\mp 2}}).$$

Further, by (3.5) we find that for $u \in C^\infty(\mathbb{H}, E_{\tau_m})$:

$$\partial_+\partial_- u = d\pi_R(\Omega) + \frac{1}{4}d\pi_R(\kappa)^2 - \frac{1}{2}id\pi_R(\kappa) = d\pi_L(\Omega) - \frac{m^2}{4} - \frac{m}{2},$$
$$\partial_-\partial_+ u = d\pi_R(\Omega) + \frac{1}{4}d\pi_R(\kappa)^2 + \frac{1}{2}id\pi_R(\kappa) = d\pi_L(\Omega) - \frac{m^2}{4} + \frac{m}{2},$$

where $d\pi_L$ is the Lie algebra representation corresponding to the left regular representation π_L. Note that $d\pi_L(\Omega) = -\widetilde{\Delta}^\sharp_{\tau_m,\rho} = -\widetilde{A}^\sharp_{\tau_m,\rho} - \frac{1}{4}$, where $\widetilde{\Delta}^\sharp_{\tau_m,\rho}$ resp. $\widetilde{A}^\sharp_{\tau_m,\rho}$ denotes the lift of $\Delta^\sharp_{\tau_m,\rho}$ resp. $A^\sharp_{\tau_m,\rho}$ to $\mathbb{H}$. Since ∂_+ and ∂_- are left-invariant, they induce operators on $C^\infty(X, E_{\tau_m,\rho})$ which, by the previous considerations, satisfy

$$\partial_+\partial_- u = -A^\sharp_{\tau_m,\rho} - \frac{(m+1)^2}{4}, \qquad \partial_-\partial_+ u = -A^\sharp_{\tau_m,\rho} - \frac{(m-1)^2}{4}. \tag{3.6}$$

Lemma 3.8.1. *For $\mu \neq -\frac{(m+1)^2}{4}$:*

$$\mathrm{mult}(\mu; A^\sharp_{\tau_m,\rho}) = \mathrm{mult}(\mu; A^\sharp_{\tau_{m+2},\rho}).$$

Proof. Since $\partial_\pm$ is given by the right regular representation and $A^\sharp_{\tau_m,\rho}$ by the left regular representation, we have

$$\partial_\pm \circ A^\sharp_{\tau_m,\rho} = A^\sharp_{\tau_{m\mp 2},\rho} \circ \partial_\pm.$$

This implies

$$\partial_\pm : L^2(X, E_{\tau_m,\rho})_\mu \to L^2(X, E_{\tau_{m\mp 2},\rho})_\mu.$$

By (3.6), the map

$$\partial_+ : L^2(X, E_{\tau_m} \otimes E_\rho)_\mu \to L^2(X, E_{\tau_{m-2},\rho})_\mu$$

is an isomorphism whenever $\mu \neq -\frac{(m-1)^2}{4}$, and the map

$$\partial_- : L^2(X, E_{\tau_m} \otimes E_\rho)_\mu \to L^2(X, E_{\tau_{m+2},\rho})_\mu$$

is an isomorphism whenever $\mu \neq -\frac{(m+1)^2}{4}$. □

Theorem 3.8.2. *For all $m \in \mathbb{R}$, we have*

$$\begin{aligned}\operatorname{mult}(-\tfrac{(m+1)^2}{4}; A^\sharp_{\tau_{m+2},\rho}) - \operatorname{mult}(-\tfrac{(m+1)^2}{4}; A^\sharp_{\tau_m,\rho}) \\ = \frac{\operatorname{Vol}(X)\dim(V_\rho)}{4\pi}(m+1) + \sum_{[\gamma]\text{ ell.}} \frac{i \operatorname{tr}\rho(\gamma)}{2M(\gamma)\sin\theta(\gamma)} e^{i(m+1)\theta(\gamma)}.\end{aligned}$$

In particular, the right-hand side is an integer.

Proof. We subtract the trace formula in Theorem 3.7.1 for m from the one for $m+2$. By Lemma 3.8.1, the left-hand side becomes

$$e^{\frac{(m+1)^2}{4}t}\big(\operatorname{mult}(-\tfrac{(m+1)^2}{4}; A^\sharp_{\tau_{m+2},\rho}) - \operatorname{mult}(-\tfrac{(m+1)^2}{4}; A^\sharp_{\tau_m,\rho})\big).$$

Carefully comparing terms on the right-hand side for m and $m+2$, we find

$$\frac{\operatorname{Vol}(X)\dim(V_\rho)}{4\pi}(m+1)e^{\frac{(m+1)^2}{4}t} + \sum_{[\gamma]\text{ ell.}} \frac{i \operatorname{tr}\rho(\gamma)}{2M(\gamma)\sin\theta(\gamma)} e^{i(m+1)\theta(\gamma)} e^{\frac{(m+1)^2}{4}t}. \quad \square$$

4 Twisted Ruelle and Selberg zeta functions

We establish the meromorphic continuation for the Selberg zeta function in Section 4.1, and a functional equation in Section 4.2. Then, we compute the behavior of the Ruelle zeta function at zero in Section 4.3, and prove Theorem A.

4.1 Meromorphic continuation

Every closed oriented geodesic γ on X lifts canonically to a homotopy class in $\widetilde{\Gamma} = \pi_1(X_1)$, still denoted by γ. The conjugacy class $[\gamma]$ in $\widetilde{\Gamma}/\widetilde{Z} \simeq \Gamma = \pi_1(X)$ consists of hyperbolic elements and conversely every such conjugacy class of hyperbolic elements contains exactly one representative of a closed geodesic.

A closed geodesic γ on X is called *prime* if it cannot be written as a multiple of a shorter geodesic.

Let $\rho : \widetilde{\Gamma} = \pi_1(X_1) \to \operatorname{GL}(V_\rho)$ be a finite-dimensional, complex representation. For $s \in \mathbb{C}$, we define the twisted Selberg zeta function

$$Z(s;\rho) = \prod_{\gamma\text{ prime}} \prod_{k=0}^{\infty} \det\left(\operatorname{Id} - \rho(\gamma)e^{-(s+k)\ell(\gamma)}\right)$$

and the twisted Ruelle zeta function

$$R(s;\rho) = \prod_{\gamma\text{ prime}} \det\left(\operatorname{Id} - \rho(\gamma)e^{-s\ell(\gamma)}\right),$$

where the products run over the prime closed geodesics in X.

It is shown in [7, Theorem 3.1] that $Z(s;\rho)$, and hence $R(s;\rho)$, converges for s in some right half plane of $\mathbb{C}$ and defines a holomorphic function on this half plane (see also [29, Section 1.2] for the torsion-free case). We note that the logarithmic derivative $L(s;\rho)$ of $Z(s;\rho)$ is given by

$$L(s;\rho) := \frac{d}{ds}\log Z(s;\rho) = \sum_{\gamma} \frac{\ell(\gamma)\operatorname{tr}(\rho(\gamma))}{2n_\Gamma(\gamma)\sinh(\frac{\ell(\gamma)}{2})} e^{-(s-\frac{1}{2})\ell(\gamma)},$$

where the summation is over *all* closed geodesics.

Moreover, we have

$$R(s;\rho) = \frac{Z(s;\rho)}{Z(s+1;\rho)}. \tag{4.1}$$

To show meromorphic continuation of $Z(s;\rho)$ and $R(s;\rho)$ to $s \in \mathbb{C}$, we let $s_1 = s \in \mathbb{C}$ and $s_2 \in \mathbb{C}$ with $\mathrm{Re}(s_1 - \frac{1}{2})^2, \mathrm{Re}(s_2 - \frac{1}{2})^2$ sufficiently large and insert the trace formula from Theorem 3.7.1 into the resolvent identity

$$\begin{aligned}\mathrm{tr}\Big(\big(A^\sharp_{\tau_m,\rho} + (s_1 - \tfrac{1}{2})^2\big)^{-1} - \big(A^\sharp_{\tau_m,\rho} + (s_2 - \tfrac{1}{2})^2\big)^{-1}\Big) \\ = \int_0^\infty \Big(e^{-t(s_1-\frac{1}{2})^2} - e^{-t(s_2-\frac{1}{2})^2}\Big)\,\mathrm{tr}(e^{-tA^\sharp_{\tau_m,\rho}})\,dt,\end{aligned}$$

which holds by the same arguments as in [8, Section 4.2] (especially the proof of [8, Lemma 4.2.2] is identical), to obtain

$$\begin{aligned}\mathrm{tr}\Big(\big(A^\sharp_{\tau_m,\rho} + (s_1 - \tfrac{1}{2})^2\big)^{-1} - \big(A^\sharp_{\tau_m,\rho} + (s_2 - \tfrac{1}{2})^2\big)^{-1}\Big) \\ = I(s_1,s_2;\rho) + H(s_1,s_2;\rho) + E(s_1,s_2;\rho),\end{aligned} \tag{4.2}$$

with the obvious meaning of the identity contribution $I(s_1,s_2;\rho)$, the hyperbolic contribution $H(s_1,s_2;\rho)$ and the elliptic contribution $E(s_1,s_2;\rho)$. For the computation of these three terms we choose $|m| \leq 1$ which is possible since m is determined by $\rho(t) = e^{-im\pi}$ and hence only determined by ρ modulo 2. Then, all contributions from the (relative) discrete series vanish.

We first compute the identity contribution

$$\begin{aligned}I(s_1,s_2;\rho) = \frac{\mathrm{Vol}(X)\dim(V_\rho)}{4\pi} \int_0^\infty \Big(e^{-t(s_1-\frac{1}{2})^2} - e^{-t(s_2-\frac{1}{2})^2}\Big) \\ \times \int_{\mathbb{R}} e^{-t\lambda^2} \frac{\lambda \sinh 2\pi\lambda}{\cosh 2\pi\lambda + \cos \pi m}\,d\lambda\,dt.\end{aligned} \tag{4.3}$$

Interchanging the integrals shows that

$$= \frac{\mathrm{Vol}(X)\dim(V_\rho)}{4\pi} \int_{\mathbb{R}} \frac{\lambda \sinh 2\pi\lambda}{\cosh 2\pi\lambda + \cos \pi m} \left(\frac{1}{\lambda^2 + (s_1 - \frac{1}{2})^2} - \frac{1}{\lambda^2 + (s_2 - \frac{1}{2})^2}\right) d\lambda.$$

This integral is computed in [12, Chapter 10, Lemma 2.4]:

$$= \frac{\mathrm{Vol}(X)\dim(V_\rho)}{4\pi} \sum_{n=0}^\infty \left[\frac{1}{n + \frac{m}{2} + s_1} + \frac{1}{n - \frac{m}{2} + s_1} - \frac{1}{n + \frac{m}{2} + s_2} - \frac{1}{n - \frac{m}{2} + s_2}\right]. \tag{4.4}$$

The hyperbolic contribution

$$\begin{aligned}H(s_1,s_2;\rho) = \int_0^\infty \Big(e^{-t(s_1-\frac{1}{2})^2} - e^{-t(s_2-\frac{1}{2})^2}\Big) \\ \times \frac{1}{2\sqrt{4\pi t}} \sum_{[\gamma]\ \mathrm{hyp.}} \frac{l(\gamma)\,\mathrm{tr}(\rho(\gamma))}{n_\Gamma(\gamma) \sinh \frac{l(\gamma)}{2}} e^{-\frac{l(\gamma)^2}{4t}}\,dt\end{aligned}$$

was calculated in [8, Proposition 4.2.4 & 4.2.5] to be

$$= \frac{L(s_1;\rho)}{2(s_1-\frac{1}{2})} - \frac{L(s_2;\rho)}{2(s_2-\frac{1}{2})}.$$

Finally, the elliptic contribution is given by

$$E(s_1,s_2;\rho) = \sum_{[\gamma]\text{ ell.}} \frac{\operatorname{tr}\rho(\gamma)}{4M(\gamma)\sin\theta(\gamma)} \int_0^\infty \left(e^{-t(s_1-\frac{1}{2})^2} - e^{-t(s_2-\frac{1}{2})^2}\right) \times \int_{\mathbb{R}} e^{-t\lambda^2} \frac{\cosh 2(\pi-\theta(\gamma))\lambda + e^{i\pi m}\cosh 2\theta(\gamma)\lambda}{\cosh 2\pi\lambda + \cos\pi m}\, d\lambda\, dt.$$

Note that summation and integration can be interchanged since the sum is finite. Interchanging the two integrals and computing the integral over t gives

$$= \sum_{[\gamma]\text{ ell.}} \frac{\operatorname{tr}\rho(\gamma)}{4M(\gamma)\sin\theta(\gamma)} \int_{\mathbb{R}} \left(\frac{1}{\lambda^2+(s_1-\frac{1}{2})^2} - \frac{1}{\lambda^2+(s_2-\frac{1}{2})^2}\right) \times \frac{\cosh 2(\pi-\theta(\gamma))\lambda + e^{i\pi m}\cosh 2\theta(\gamma)\lambda}{\cosh 2\pi\lambda + \cos\pi m}\, d\lambda.$$

The integral over λ is computed in [12, Chapter 10, Lemma 2.7]:

(4.5)

$$= \sum_{[\gamma]\text{ ell.}} \frac{\operatorname{tr}\rho(\gamma)}{4M(\gamma)\sin\theta(\gamma)} \left[\frac{i}{s_1-\frac{1}{2}} \sum_{n=0}^\infty \left(\frac{e^{-2i\theta(\gamma)(n-\frac{m}{2}+\frac{1}{2})}}{n-\frac{m}{2}+s_1} - \frac{e^{2i\theta(\gamma)(n+\frac{m}{2}+\frac{1}{2})}}{n+\frac{m}{2}+s_1}\right) - \frac{i}{s_2-\frac{1}{2}} \sum_{n=0}^\infty \left(\frac{e^{-2i\theta(\gamma)(n-\frac{m}{2}+\frac{1}{2})}}{n-\frac{m}{2}+s_2} - \frac{e^{2i\theta(\gamma)(n+\frac{m}{2}+\frac{1}{2})}}{n+\frac{m}{2}+s_2}\right)\right].$$

Proposition 4.1.1. *The logarithmic derivative $L(s;\rho)$ of the twisted Selberg zeta function $Z(s;\rho)$ has a meromorphic extension to the whole complex plane $\mathbb{C}$. Its singularities are given by the following formal expression:*

$$\sum_{j=0}^\infty \left[\frac{1}{s-\frac{1}{2}-i\mu_j} + \frac{1}{s-\frac{1}{2}+i\mu_j}\right] + \frac{\operatorname{Vol}(X)\dim(V_\rho)}{4\pi} \sum_{n=0}^\infty \left[\frac{1-2s}{s+\frac{m}{2}+n} + \frac{1-2s}{s-\frac{m}{2}+n}\right] + i \sum_{[\gamma]\text{ ell.}} \frac{\operatorname{tr}\rho(\gamma)}{2M(\gamma)\sin\theta(\gamma)} \sum_{n=0}^\infty \left[\frac{e^{2i\theta(\gamma)(n+\frac{m}{2}+\frac{1}{2})}}{s+\frac{m}{2}+n} - \frac{e^{-2i\theta(\gamma)(n-\frac{m}{2}+\frac{1}{2})}}{s-\frac{m}{2}+n}\right]$$

where $(\frac{1}{4}+\mu_j^2)_{j\in\mathbb{Z}_{\geq 0}} \subseteq \mathbb{C}$ are the eigenvalues of $\Delta^\sharp_{\tau_m,\rho}$, counted with algebraic multiplicity.

Proof. Fix $s_2 \in \mathbb{C}$ with $\operatorname{Re}(s_2-\frac{1}{2})^2$ sufficiently large and let $s = s_1 \in \mathbb{C}$. Multiplying (4.2) by $2(s_1-\frac{1}{2})$ gives

$$L(s;\rho) = 2(s-\tfrac{1}{2}) \operatorname{tr}\left(\left(A^\sharp_\rho + (s-\tfrac{1}{2})^2\right)^{-1} - \left(A^\sharp_\rho + (s_2-\tfrac{1}{2})^2\right)^{-1}\right) - 2(s-\tfrac{1}{2})I(s,s_2;\rho) - 2(s-\tfrac{1}{2})E(s,s_2;\rho) + \frac{s-\frac{1}{2}}{s_2-\frac{1}{2}} L(s_2;\rho).$$

Together with (4.4) and (4.5), this shows that $L(s;\rho)$ extends meromorphically to all $s \in \mathbb{C}$. The contribution of the first two terms to the poles of $L(s;\rho)$ was obtained in [8, Proposition 4.2.5]. The last term is holomorphic in s, and for the term $E(s, s_2; \rho)$ one can use (4.5). □

Theorem 4.1.2 (Meromorphic continuation of the Selberg zeta function)**.** *The twisted Selberg zeta function $Z(s;\rho)$ has a meromorphic extension to the whole complex plane $\mathbb{C}$ with poles and zeros contained in the set $\{\frac{1}{2} \pm i\mu_j : j \in \mathbb{Z}_{\geq 0}\} \cup \{\pm\frac{m}{2} - n : n \in \mathbb{Z}_{\geq 0}\}$, where $(\frac{1}{4} + \mu_j^2)_{j \in \mathbb{Z}_{\geq 0}} \subseteq \mathbb{C}$ are the eigenvalues of $\Delta^{\sharp}_{\tau_m,\rho}$.*

Proof. In view of Proposition 4.1.1 it suffices to show that the residues of $L(s;\rho)$ are integers. For the poles at $s = \frac{1}{2} \pm i\mu_j$ the residue is the algebraic multiplicity of μ_j. For $s = \mp\frac{m}{2} - n$ the residue is

$$\frac{\operatorname{Vol}(X)\dim(V_\rho)}{4\pi}(\pm m + 2n + 1) \pm i \sum_{[\gamma] \text{ ell.}} \frac{\operatorname{tr}\rho(\gamma)}{2M(\gamma)\sin\theta(\gamma)} e^{\pm 2i\theta(\gamma)(n \pm \frac{m}{2} + \frac{1}{2})}$$

which is an integer by Theorem 3.8.2. □

Remark 4.1.3. Since $\Delta^{\sharp}_{\tau_m,\rho}$ is in general not self-adjoint, its eigenvalues might be complex, although contained in a positive cone in $\mathbb{C}$. It may therefore happen that finitely many of the singularities $\frac{1}{2} \pm i\mu_j$ of $L(s;\rho)$ coincide with finitely many of the singularities $\pm\frac{m}{2} - n$. In particular, we are not able to decide more precisely what are the poles and the zeros of the Selberg zeta function in Proposition 4.1.1.

4.2 The functional equation

From the trace formula we further conclude:

Theorem 4.2.1 (Functional equation)**.** *The twisted Selberg zeta function $Z(s;\rho)$ satisfies the following functional equation:*

$$\eta(s;\rho) = \frac{Z(s;\rho)}{Z(1-s;\rho)} = \exp\left(\int_0^{s-\frac{1}{2}} \dim(V_\rho)\operatorname{Vol}(X) \frac{\xi \sin 2\pi\xi}{\cos 2\pi\xi + \cos \pi m} \right.$$
$$\left. - \sum_{[\gamma] \text{ ell.}} \frac{\pi \operatorname{tr}\rho(\gamma)}{M(\gamma)\sin\theta(\gamma)} \frac{\cos 2(\pi - \theta(\gamma))\xi + e^{i\pi m}\cos 2\theta(\gamma)\xi}{\cos 2\pi\xi + \cos \pi m}\, d\xi \right),$$

where the integral is along any contour from 0 to $s - \frac{1}{2}$ in the complex plane avoiding the zeros of the denominator.

Proof. Fix $s_2 \in \mathbb{C}$ with $\operatorname{Re}(s_2 - \frac{1}{2})^2$ sufficiently large and subtract from (4.2) with $s_1 = s$ the same identity with $s_1 = 1 - s$ instead. This yields

$$0 = I(s, 1-s;\rho) + E(s, 1-s;\rho) + H(s, 1-s;\rho).$$

The identity contribution equals

$$I(s, 1-s;\rho) = -\frac{\dim(V_\rho)\operatorname{Vol}(X)}{4}\big(\tan \pi(s - \tfrac{1}{2} + \tfrac{m}{2}) + \tan \pi(s - \tfrac{1}{2} - \tfrac{m}{2})\big),$$

by [8, proof of Theorem 4.2.8]. Using

$$\tan(x+y)+\tan(x-y)=\frac{2\sin(2x)}{\cos(2x)+\cos(2y)},$$

this becomes

$$I(s,1-s;\rho)=-\frac{\dim(V_\rho)\operatorname{Vol}(X)}{2}\frac{\sin 2\pi(s-\frac{1}{2})}{\cos 2\pi(s-\frac{1}{2})+\cos\pi m}.$$

The hyperbolic contribution becomes

$$H(s,1-s;\rho)=\frac{L(s;\rho)+L(1-s;\rho)}{2(s-\frac{1}{2})}$$

and the elliptic contribution is

$$\begin{aligned}E(s,1-s;\rho)&=\sum_{[\gamma]\text{ ell.}}\frac{\operatorname{tr}\rho(\gamma)}{4M(\gamma)\sin\theta(\gamma)}\left[\frac{i}{s-\frac{1}{2}}\sum_{n\in\mathbb{Z}}\left(\frac{e^{2i\theta(\gamma)(n+\frac{m}{2}+\frac{1}{2})}}{s-\frac{m}{2}-n-1}-\frac{e^{2i\theta(\gamma)(n+\frac{m}{2}+\frac{1}{2})}}{s+\frac{m}{2}+n}\right)\right]\\&=\sum_{[\gamma]\text{ ell.}}\frac{\pi\operatorname{tr}\rho(\gamma)}{2M(\gamma)(s-\frac{1}{2})\sin\theta(\gamma)}\frac{\cos 2(\pi-\theta(\gamma))(s-\frac{1}{2})+e^{i\pi m}\cos 2\theta(\gamma)(s-\frac{1}{2})}{\cos 2\pi(s-\frac{1}{2})+\cos\pi m}\end{aligned}$$

by [12, equation (**) on page 444]. This implies

$$\begin{aligned}\frac{d}{ds}\log\frac{Z(s;\rho)}{Z(1-s;\rho)}&=L(s;\rho)+L(1-s;\rho)\\&=\dim(V_\rho)\operatorname{Vol}(X)(s-\tfrac{1}{2})\frac{(s-\frac{1}{2})\sin 2\pi(s-\frac{1}{2})}{\cos 2\pi(s-\frac{1}{2})+\cos\pi m}\\&\quad-\sum_{[\gamma]\text{ ell.}}\frac{\pi\operatorname{tr}\rho(\gamma)}{M(\gamma)\sin\theta(\gamma)}\frac{\cos 2(\pi-\theta(\gamma))(s-\frac{1}{2})+e^{i\pi m}\cos 2\theta(\gamma)(s-\frac{1}{2})}{\cos 2\pi(s-\frac{1}{2})+\cos\pi m}\end{aligned}\tag{4.6}$$

and the claim follows by integration and exponentiation. □

4.3 Fried's conjecture

We compute the behavior of the Ruelle zeta function at $s=0$, and we prove Theorem A.

We start with the following proposition:

Proposition 4.3.1. *The Ruelle zeta function extends meromorphically to the whole complex plane. Moreover,*

$$R(s;\rho)\sim_{s\to 0}\pm\eta(s+1;\rho)^{-1}$$

with $\eta(s;\rho)$ as in Theorem 4.2.1.

Proof. Using Eq. (4.1) and Theorem 4.1.2 we directly deduce that $R(s;\rho)$ extends meromorphically. Together with the functional equation for the Selberg zeta function established in Theorem 4.2.1, we get

$$R(s;\rho)=\frac{Z(s;\rho)}{Z(1-s;\rho)}=\frac{Z(s;\rho)}{\eta(1+s;\rho)Z(-s;\rho)}$$

and the result follows. □

Now, we are led to compute the value of $\eta(s+1;\rho)$ at $s=0$. The rest of this section is devoted to this computation. We summarize the result in the following theorem. Recall that each singular point in the orbisurface X is represented by an elliptic loop $c_j \in \pi_1(X_1)$ of finite order ν_j. For any $j=1,\dots,s$ we denote by $n_j = \dim \operatorname{Fix} \rho(c_j)$, so that $\rho(c_j) = I_{n_j} \oplus T_j$. Let $M_j(x) = \det(I_{n-n_j} - T_j)(\frac{ix}{\nu_j})^{n_j}$ and $M_0(x) = (ix)^n$ for $x \in \mathbb{C}$.

Theorem 4.3.2. *For $\rho\colon \pi_1(X_1) \to \mathrm{GL}(V)$ an irreducible representation:*

1. *If $m=0$, then*
$$R\left(\frac{x}{2\pi},\rho\right) \sim_{x\to 0} \pm \frac{M_0(x)^{2g-2+s}}{\prod_{j=1}^s M_j(x)}.$$
2. *If $m \neq 0$, then*
$$R(0;\rho) = \pm \frac{\det(\rho(t)-I_n)^{2g+s-2}}{\prod_{j=1}^s \det(\rho(c_j)-I_n)}.$$

Now we prove Theorem 4.3.2 by computing the behavior of $\eta(1+s;\rho)$ near $s=0$. We start with the following observation:

Lemma 4.3.3.

$$\eta(1+s;\rho) = \exp \int_0^{s+\frac{1}{2}} \frac{\eta'(\xi+\frac{1}{2};\rho)}{\eta(\xi+\frac{1}{2};\rho)} d\xi \tag{4.7}$$

Remark 4.3.4. Note that the previous expression makes sense independently on the chosen path between 0 and $s+\frac{1}{2}$, since the residues of the logarithmic derivative of the meromorphic function η are in $2i\pi\mathbb{Z}$.

Proof. Note that the equality holds true at $s=-\frac{1}{2}$, where indeed both side are equal to 1. Let $s\in\mathbb{C}$ such that $\eta(1+s;\rho)$ is neither 0 nor ∞. Then, taking a determination of the logarithm and differentiating both sides, we find that both sides of the equation locally coincide, up to a multiplicative constant. Since both sides define a meromorphic function, the constant should be global. Considering $s=-\frac{1}{2}$, we conclude that it is equal to 1. ☐

In order to compute the integral in Eq. (4.7), we regroup the terms as in Eq. (4.6):

$$\frac{\eta'(\xi+\frac{1}{2};\rho)}{\eta(\xi+\frac{1}{2};\rho)} = \dim(V_\rho)\operatorname{Vol}(X)\frac{-2\pi\xi\sin 2\pi\xi}{\cos 2\pi\xi + \cos\pi m} + \sum_j E_j(\xi), \tag{4.8}$$

where

$$E_j(\xi) = -\sum_{k=1}^{\nu_j-1} \frac{\pi \operatorname{tr}\rho(c_j^k)}{\nu_j \sin(k\pi/\nu_j)} \frac{\cos\left(2(\pi - k\pi/\nu_j)\xi\right) + e^{i\pi m}\cos(2k\pi\xi/\nu_j)}{\cos 2\pi\xi + \cos\pi m} \tag{4.9}$$

regroups all the summands corresponding to the elliptic element c_j and its powers.

In view of Remark 4.3.4, one needs to make a choice of an integration path avoiding the poles of the integrand in Eq. (4.7). The integrand has a unique pole in $[0,\frac{1}{2}]$, which we denote by ξ_0 (its value is $\frac{1-m}{2} \mod \mathbb{Z}$). We choose a path from 0 to $\frac{1}{2}$ which avoids ξ_0 along a small half-circle in the upper-half plane and we denote this path by $\mathscr{C}_+$, oriented following the increasing real part. Let $\mathscr{C}_- = \overline{\mathscr{C}_+}$, then $\mathscr{C} = \mathscr{C}_+ - \mathscr{C}_-$ is a small circle encircling ξ_0.

First, we compute the contribution of the identity, namely the integral of the first summand in the right-hand side of (4.8).

Lemma 4.3.5.

$$\exp\left(\int_0^{1/2} \frac{-2\pi\xi \sin 2\pi\xi d\xi}{\cos 2\pi\xi + \cos \pi m}\right) = \pm(1 - e^{i\pi m}) \qquad \textit{for } m \neq 0, \tag{4.10}$$

$$\exp\left(\int_0^{1/2+s} \frac{-2\pi\xi \sin 2\pi\xi d\xi}{\cos 2\pi\xi + 1}\right) \sim_{s=0} -2\pi s + o(s) \qquad \textit{for } m = 0.$$

Proof. The asymptotic case (for $m = 0$) goes through exactly as in [10, Lemma 3], so we just prove the Eq. (4.10) (for $m \neq 0$). By [10, Lemma 2], we know that

$$\mathrm{Re}\left(\int_0^{1/2} \frac{-2\pi\xi \sin 2\pi\xi d\xi}{\cos 2\pi\xi + \cos \pi m}\right) = \log|1 - e^{i\pi m}|.$$

Hence, we only need to show that

$$\mathrm{Im}\left(\int_0^{1/2} \frac{-2\pi\xi \sin 2\pi\xi d\xi}{\cos 2\pi\xi + \cos \pi m}\right) = \arg(1 - e^{i\pi m}) \mod \pi,$$

where for the integral we mean $\int_{\mathscr{C}_+} \frac{-2\pi\xi \sin 2\pi\xi d\xi}{\cos 2\pi\xi + \cos \pi m}$.

Let $f(\xi) = \cos 2\pi\xi + \cos \pi m$. We want to compute the imaginary part of the integral

$$\int_{\mathscr{C}_+} \xi \frac{f'(\xi)}{f(\xi)} d\xi.$$

Note that the function $\xi \mapsto \xi \frac{f'(\xi)}{f(\xi)}$ sends the real axis to itself, hence we have $\int_{\mathscr{C}_-} \xi \frac{f'(\xi)}{f(\xi)} d\xi = \overline{\int_{\mathscr{C}_+} \xi \frac{f'(\xi)}{f(\xi)} d\xi}$ and $\mathrm{Im}\left(\int_{\mathscr{C}_-} \xi \frac{f'(\xi)}{f(\xi)} d\xi\right) = -\mathrm{Im}\left(\int_{\mathscr{C}_+} \xi \frac{f'(\xi)}{f(\xi)} d\xi\right)$. We deduce that

$$\mathrm{Im}\left(\int_{\mathscr{C}_+} \xi \frac{f'(\xi)}{f(\xi)} d\xi\right) = \frac{1}{2}\mathrm{Im}\left(\int_{\mathscr{C}} \xi \frac{f'(\xi)}{f(\xi)} d\xi\right) = \pi \,\mathrm{Res}\left(\xi \frac{f'(\xi)}{f(\xi)}, \xi_0\right) = \pi\xi_0.$$

A computation shows $\arg((1-e^{i\pi m})^2) = \pi(1-m) \mod 2\pi$, hence $\arg(1-e^{i\pi m}) = \pi\xi_0 \mod \pi$ and Eq. (4.10) follows. ☐

Now, we compute the terms coming from the elliptic elements. We first treat the case $m \neq 0$, which generalizes [10, Lemma 5]:

Lemma 4.3.6. *For $m \neq 0$ we have*

$$\exp \int_0^{1/2} E_j = \pm \frac{\det(I_n - \rho(c_j))}{\det(I_n - \rho(t))^{1/\nu_j}}. \tag{4.11}$$

Proof. In [10, Lemma 5], it is shown that the equality Eq. (4.11) holds true in moduli. Following the lines of [10, Proof of Lemma 5], we compute the arguments of the terms involved. For each i, recall that the contribution E_j of the elliptic element c_j and its powers is given in Eq. (4.9). The term $\operatorname{Tr}\rho(c_j)^k$ is $\sum_\alpha \alpha^k$, for α running through the eigenvalues of $\rho(c_j)$. In particular $\alpha^M = e^{-i\pi m}$, where $M = \nu_j$ is the order of c_j (recall that $\rho(t) = e^{-i\pi m}$). We compute the contribution of each α to $\int_{\mathscr{C}_+} E_j$ as follows: we set $z_0 = e^{i\pi/M}$ and $z = e^{2i\pi\xi/M}$ and get that α contributes $\int_{\mathcal{C}_+} \frac{P(z)}{Q(z)}dz$, where

$$
\begin{aligned}
P(z) &= -\frac{1}{2iz}\sum_{k=1}^{M-1} \frac{\alpha^k}{\sin(k\pi/M)}(z^{M-k} + z^{k-M} + e^{i\pi m}(z^k + z^{-k}))z^M,\\
Q(z) &= (z^M + e^{i\pi m})(z^M + e^{-i\pi m})
\end{aligned}
$$

are polynomials in z, and $\mathcal{C}_+$ is the image of the path $\mathscr{C}_+$, by the map $\xi \mapsto e^{2i\pi\xi/M}$. It is a small deformation of an arc of the unit circle, avoiding the pole occurring at $e^{2i\pi\xi_0/M}$.

Indeed, one can rearrange P by grouping the terms corresponding to k and $M-k$ in the sum, factorizing $(z^k + z^{-k})$ and re-expanding:

$$
P(z) = -\frac{1}{2iz}\sum_{k=1}^{M-1} \frac{\alpha^k e^{i\pi m} + \alpha^{-k}e^{-i\pi m}}{\sin(k\pi/M)}(z^k + z^{-k})z^M.
$$

Let us assume that $m \neq \pm 1$, so that the polynomial Q splits as the product $Q(z) = \prod_{l=1}^{2M}(z - z_l)$ with z_l simple roots. Hence, we can decompose the rational fraction P/Q as

$$
\frac{P(z)}{Q(z)} = \sum_{l=1}^{2M} \frac{P(z_l)/Q'(z_l)}{z - z_l}.
$$

Since $m \notin \mathbb{Z}$, one can integrate to obtain

$$
\int_{\mathcal{C}_+} \frac{P(z)}{Q(z)}dz = \sum_{l=1}^{2M} \frac{P(z_l)}{Q'(z_l)} \log\left(\frac{z_0 - z_l}{1 - z_l}\right), \tag{4.12}
$$

keeping in mind that the final result will not depend on the chosen determination of the logarithm.

Now

$$
\frac{P(z_l)}{Q'(z_l)} = \sum_{k=1}^{M-1} \frac{(\alpha^k e^{i\pi m} + \alpha^{-k}e^{-i\pi m})(z_l^k + z_l^{-k})}{2iM\varepsilon(e^{i\pi m} - e^{-i\pi m})\sin(k\pi/M)},
$$

where $\varepsilon = \pm 1$ depending on $z_l^M = e^{\varepsilon i\pi m}$. In particular, this term is real, and is the same for both z_l and z_l^{-1}. Hence, the contribution of z_l and z_l^{-1} together to Eq. (4.12) equals $P(z_l)/Q'(z_l)$ times

$$
\begin{aligned}
\log\left(\frac{z_0 - z_l}{1 - z_l}\right) - \log\left(\frac{z_0 - z_l^{-1}}{1 - z_l^{-1}}\right) &= \log(z_0) + \log(1 - \frac{z_l}{z_0}) - \log(1 - z_l z_0)\\
&= \frac{i\pi}{M} + \log(1 - \frac{z_l}{z_0}) - \log(1 - z_l z_0).
\end{aligned}
$$

All together, setting $\omega = \frac{z_l}{z_0}$ for z_l with $\varepsilon = 1$, and $\omega = z_l z_0$ for the corresponding z_l^{-1}, we obtain:

$$\begin{aligned}\int_{\mathcal{C}_+} \frac{P(z)}{Q(z)} dz &= i\pi + \sum_{\omega^M = e^{-i\pi m}} \log(1-\omega) \sum_{k=1}^{M-1} \frac{(\alpha^k e^{i\pi m} + \alpha^{-k} e^{-i\pi m})}{2iM(e^{i\pi m} - e^{-i\pi m}) \sin(k\pi/M)} \\ &\qquad \times (z_0^k \omega^k + z_0^{-k} \omega^{-k} - z_0^{-k} \omega^k - z_0^k \omega^{-k}) \\ &= i\pi + \sum_{\omega^M = e^{-i\pi m}} \log(1-\omega) \sum_{k=1}^{M-1} \frac{(\alpha^k e^{i\pi m} + \alpha^{-k} e^{-i\pi m})}{M(e^{i\pi m} - e^{-i\pi m})} (\omega^k - \omega^{-k}).\end{aligned}$$

Note that $\omega = \zeta\alpha$, with $\zeta^M = 1$. In the sum above, the term for k simplifies with the term for $M-k$ and one gets

$$\frac{1}{M(e^{i\pi m} - e^{-i\pi m})} \sum_{k=1}^{M-1} e^{-i\pi m}\zeta^k - e^{i\pi m}\zeta^{-k} = \begin{cases} -1/M \text{ if } \zeta \neq 1 \\ (M-1)/M \text{ if } \zeta = 1 \end{cases}.$$

Hence, we obtain

$$\int_{\mathcal{C}_+} \frac{P}{Q} = i\pi + \log(1-\alpha) - \frac{1}{M} \sum_{\omega^M = e^{-i\pi m}} \log(1-\omega).$$

The last sum above is just $\log(1 - e^{-i\pi m})$. Adding all this for every α and every index i, and taking the exponential, we obtain Eq. (4.11).

For $m = \pm 1$, one needs to be a bit more careful when decomposing $Q(z)$. However, the computation goes through, combining what we have just done and [10, Section 3]. Hence, we omit this case. □

Now we generalize [10, Lemma 6], which computes the elliptic contribution in the case $m = 0$:

Lemma 4.3.7. *For $m = 0$ we have*

(4.13)
$$\exp \int_0^{1/2-\varepsilon} E_j \sim_{\varepsilon\to 0} \pm(\det T_j)^{-\frac{1}{2}} \det(I_{n-n_j} - T_j)(2\pi\varepsilon)^{-\frac{n}{\nu_j}+n_j}\, i^{n-n_j} \nu_j^{-n_j}.$$

Proof. We follow [10, Proof of Lemma 6] and compute in each step the phase factors involved. Let α, z_0, P and Q be as in the proof of Lemma 4.3.6. By the same arguments as in [10, proof of Lemma 6], we can write

$$\frac{P(z)}{Q(z)} = \sum_{z_1^M = -1} \frac{\mu(z_1)}{z - z_1},$$

with $z = e^{2\pi i\xi/M}$. The residues $\mu(z_1)$ are computed in [10, Lemma 7] and are given by

$$\mu(z_0^{2k+1}) = v(k) - \frac{2k+1}{M} \qquad (0 \leq k \leq M-1),$$

where $v(k) = \#\{x \in \mathbb{Z} : |x| \leq k, x \equiv p \mod M\}$ with p chosen such that $\alpha = z_0^{2p}$. For $z_1 \neq z_0$ we find

$$\int_0^{\frac{1}{2}} \frac{\mu(z_1)}{z - z_1} d\xi = \int_1^{z_0} \frac{\mu(z_1)}{z - z_1} dz = \mu(z_1) \log\left(\frac{z_0 - z_1}{1 - z_1}\right),$$

and for $z_1 = z_0$ we let $z_\varepsilon = z_0 e^{-2\pi i\varepsilon/M}$ and get

$$\int_0^{\frac{1}{2}-\varepsilon} \frac{\mu(z_0)}{z - z_0} d\xi = \int_1^{z_\varepsilon} \frac{\mu(z_0)}{z - z_0} dz = \mu(z_0) \log\left(\frac{z_\varepsilon - z_0}{1 - z_0}\right).$$

Remark 4.3.8. Note once again that the final result does not depend on the chosen determination of the logarithm. In particular the residues $\mu(z_0^k)$ are rational numbers in $\frac{1}{M}\mathbb{Z}$, hence the values of the complex numbers computed in the sequel such as $\left(\frac{z_\varepsilon - z_0}{1-z_0}\right)^{\mu(z_0)}$ or $\left(\frac{z_0 - z_1}{1-z_1}\right)^{\mu(z_1)}$ are only well-defined up to a (fixed) M-th root of unity. Since at the end, we take the product of M such factors, this ambiguity cancels out.

Hence, the total contribution of the eigenvalue α to $\exp \int_0^{\frac{1}{2}-\varepsilon} E_j(\xi)\, d\xi$ amounts to

$$\begin{aligned} \exp \int_0^{\frac{1}{2}-\varepsilon} \frac{P(z)}{Q(z)} d\xi &= \left(\frac{z_\varepsilon - z_0}{1 - z_0}\right)^{\mu(z_0)} \prod_{\substack{z_1^M = -1 \\ z_1 \neq z_0}} \left(\frac{z_0 - z_1}{1 - z_1}\right)^{\mu(z_1)} \\ &\sim \left(\frac{-2\pi i\varepsilon}{M(1 - z_0)}\right)^{\mu(z_0)} \prod_{\substack{z_1^M = -1 \\ z_1 \neq z_0}} \left(\frac{1 - z_1/z_0}{1 - z_1}\right)^{\mu(z_1)}, \end{aligned} \tag{4.14}$$

where we have used $e^{-2\pi i\varepsilon/M} - 1 \sim -2\pi i\varepsilon/M$ and the fact that $\sum_{z_1} \mu(z_1) = 0$ by the Residue Theorem. To compute the product, we generalize [10, Lemma 8 and Lemma 9].

We first claim that the following generalization of [10, Lemma 8] holds:

Claim.

$$\frac{1}{M(1 - z_0)} \prod_{k=1}^{M-1} \left(\frac{1 - z_0^{2k}}{1 - z_0^{2k+1}}\right)^{2k+1} = (-i)^{M-1} \left(\frac{M}{2}\right)^M. \tag{4.15}$$

Proof of the claim. In the numerator we can combine the factor for k with the one for $M - k$ to find

$$(1-z_0^{2k})^{2k+1}(1-z_0^{2M-2k})^{2M-2k+1} = (-1)^M z_0^{(2k)^2}(1-z_0^{2k})^{M+1}(1-z_0^{2M-2k})^{M+1},$$

where we have used that $(1 - z_0^{2k}) = -z_0^{2k}(1 - z_0^{2M-2k})$. This results in

$$\prod_{k=1}^{M-1} (1 - z_0^{2k})^{2k+1} = \prod_{k=1}^{\lfloor \frac{M-1}{2} \rfloor} (-1)^M z_0^{(2k)^2} \prod_{k=1}^{M-1} (1 - z_0^{2k})^{M+1}. \tag{4.16}$$

Similarly, in the denominator we rewrite

$$\begin{aligned} (1 - z_0^{2k+1})^{2k+1}&(1 - z_0^{2M-2k-1})^{2M-2k-1} \\ &= (-1)^M z_0^{(2k+1)^2}(1 - z_0^{2k+1})^M (1 - z_0^{2M-2k-1})^M \end{aligned}$$

to find

$$\prod_{k=0}^{M-1} (1 - z_0^{2k+1})^{2k+1} = \prod_{k=0}^{\lfloor \frac{M-2}{2} \rfloor} (-1)^M z_0^{(2k+1)^2} \prod_{k=0}^{M-1} (1 - z_0^{2k+1})^M. \tag{4.17}$$

Note that the sign $(-1)^M$ in (4.16) and (4.17) is either $+1$ for M even or occurs the same number of times for M odd. Therefore, combining (4.16) and (4.17) yields

$$\frac{\prod_{k=1}^{M-1}(1-z_0^{2k})^{2k+1}}{\prod_{k=0}^{M-1}(1-z_0^{2k+1})^{2k+1}} = z_0^K \frac{\prod_{k=1}^{M-1}(1-z_0^{2k})^{M+1}}{\prod_{k=0}^{M-1}(1-z_0^{2k+1})^{M}},$$

where

$$K = \sum_{k=1}^{\lfloor\frac{M-1}{2}\rfloor}(2k)^2 - \sum_{k=1}^{\lfloor\frac{M}{2}\rfloor}(2k-1)^2 = \frac{M(M-1)}{2} \times \begin{cases} +1 & \text{for } M \text{ odd}, \\ -1 & \text{for } M \text{ even}. \end{cases}$$

Hence,

$$z_0^K = e^{\pm\frac{i\pi(M-1)}{2}} = (\pm i)^{M-1} = (-i)^{M-1},$$

where we have used that $i^{M-1} = (-i)^{M-1}$ for M odd. Finally, (4.15) follows after evaluating the remaining products using the fact that $\prod_{i=1}^{q-1} 1 - y^i = q$ for any primitive q-th root of unity. □

Next, we claim that the following generalization of [10, Lemma 9] holds:

Claim.

$$\frac{\prod_{k=1}^{M-1}(1-z_0^{2k})^{v(k)}}{\prod_{k=0}^{M-1}(1-z_0^{2k+1})^{v(k)}} = \frac{M}{2} \times \begin{cases} 1 & \textit{for } \alpha = 1, \\ \pm\alpha^{-\frac{1}{2}}(1-\alpha) & \textit{for } \alpha \neq 1. \end{cases} \tag{4.18}$$

Proof of the claim. To show this, first observe that for $|p| \leq \frac{M}{2}$:

$$v(k) = \begin{cases} 0 & \text{for } k < |p|, \\ 1 & \text{for } |p| \leq k < M - |p|, \\ 2 & \text{for } M - |p| \leq k < M. \end{cases}$$

Then one can again combine terms in the products as above to find that the left hand side in (4.18) equals

$$\frac{\prod_{k=1}^{M-1}(1-z_0^{2k})}{\prod_{k=0}^{M-1}(1-z_0^{2k+1})} \times \begin{cases} 1 & \text{for } p = 0, \\ (-z_0^{|p|})(1-z_0^{-2|p|}) & \text{for } p \neq 0. \end{cases}$$

Evaluating the products as before and rewriting

$$(-z_0^{|p|})(1-z_0^{-2|p|}) = \pm z_0^{-p}(1-z_0^{2p})$$

shows (4.18), using $z_0^{2p} = \alpha$. □

Now we are back to the proof of Lemma 4.3.7. We insert (4.15) and (4.18) into (4.14), but we need to distinguish the case $\alpha = 1$ from the case $\alpha \neq 1$. Indeed, if $\alpha = 1$, one finds $v(0) = 1$, while $v(0) = 0$ if $\alpha \neq 1$.

Now for each α, the contribution to $\int_0^{\frac{1}{2}-\varepsilon} E_j$ equals

$$E_j(\alpha) = \begin{cases} \pm\alpha^{-1/2}(1-\alpha)(2i\pi\varepsilon)^{-\frac{1}{M}}\, i^{1-\frac{1}{M}} & \text{for } \alpha \neq 1, \\ \pm(2i\pi\varepsilon)^{1-\frac{1}{M}}\, i^{1-\frac{1}{M}} M^{-1} & \text{for } \alpha = 1. \end{cases}$$

Thanks to Remark 4.3.8 we can deal with the M-th roots of i involved in (4.14), in particular they cancel out in the case $\alpha = 1$. We obtain

$$\begin{aligned}\int_0^{\frac{1}{2}-\varepsilon} E_j &= E_j(1)^{n_j} \prod_{\alpha\neq 1} E_j(\alpha) \\ &= \pm \left(\frac{(2\pi\varepsilon)^{1-\frac{1}{M}}}{M} \right)^{n_j} \prod_{\alpha\neq 1} \alpha^{-\frac{1}{2}}(1-\alpha)(2i\pi\varepsilon)^{-\frac{1}{M}}\, i^{1-\frac{1}{M}} \\ &= \det(T_j)^{-\frac{1}{2}} \det(I_{n-n_j} - T_j)(2\pi\varepsilon)^{n_j - \frac{n}{M}}\, i^{n-n_j} M^{-n_j}\end{aligned}$$

Replacing M by ν_j finishes the proof of Lemma 4.3.7. □

Proof of Theorem 4.3.2. First we assume $m = 0$. Then $\rho(t) = I_n$, and Theorem 4.3.2 (1) reads

$$R\left(\frac{x}{2\pi};\rho\right) \sim_{x\to 0} \frac{(ix)^{n(2g-2+s)-\sum_{j=1}^s n_j}}{\prod_{j=1}^s \det(I_{n-n_j} - T_j)\nu_j^{n_j}} + o(x).$$

Now combining the second statement of Lemma 4.3.5 with Lemma 4.3.7 yields

$$\begin{aligned}\eta\left(1+\frac{x}{2\pi};\rho\right) &\sim_{x\to 0} \pm x^{\dim(V_\rho)\chi(X)} \\ &\times \prod_{j=1}^s (\det T_j)^{-\frac{1}{2}} \det(I_{n-n_j} - T_j) x^{-\dim(V_\rho)/\nu_j + n_j}\, i^{n-n_j} \nu_j^{-n_j} + o(x).\end{aligned}$$

Recall that $n = \dim(V_\rho)$, $-\chi(X) = 2g-2+s-\sum_i \frac{1}{\nu_j}$ and that

$$\prod_{j=1}^s \det T_j = \prod_{j=1}^s \det \rho(c_j) = \det \rho(t)^{2g-2+s} = 1$$

by (1.1). Moreover, observe that

$$\prod_{j=1}^s i^{n-n_j} = i^{ns-\sum_j n_j} = \pm i^{n(2g-2+s)-\sum_{j=1}^s n_j}$$

and the result follows.

Now if $m \neq 0$, we directly deduce Theorem 4.3.2 (2) from the first statement in Lemma 4.3.5 and Lemma 4.3.6, since $\rho(t) = e^{-i\pi m} I_n$. □

References

[1] Dan Barbasch and Henri Moscovici, *L^2-index and the Selberg trace formula*, J. Funct. Anal. **53** (1983), no. 2, 151–201.

[2] Ulrich Bunke and Martin Olbrich, *Selberg zeta and theta functions. A differential operator approach*, Mathematical Research, vol. 83, Akademie-Verlag, Berlin, 1995.

[3] Thomas A. Chapman, *Topological invariance of Whitehead torsion*, Amer. J. Math. **96** (1974), 488–497.

[4] Yann Chaubet and Nguyen Viet Dang, *Dynamical torsion for contact Anosov flows*, (2019), preprint, available at arXiv:1911.09931.

[5] Nguyen Viet Dang, Colin Guillarmou, Gabriel Rivière, and Shu Shen, *The Fried conjecture in small dimensions*, Invent. Math. **220** (2020), no. 2, 525–579.

[6] Ksenia Fedosova, *The twisted Selberg trace formula and the Selberg zeta function for compact orbifolds*, (2015), preprint, available at arXiv:1511.04208.

[7] Ksenia Fedosova and Anke Pohl, *Meromorphic continuation of Selberg zeta functions with twists having non-expanding cusp monodromy*, Selecta Math. (N.S.) **26** (2020), no. 1, Paper No. 9.

[8] Jan Frahm and Polyxeni Spilioti, *Twisted dynamical zeta functions on compact hyperbolic surfaces*, (2021), preprint, available at arXiv:2105.13321.

[9] David Fried, *Analytic torsion and closed geodesics on hyperbolic manifolds*, Invent. Math. **84** (1986), 523–540.

[10] ———, *Fuchsian groups and Reidemeister torsion*, The Selberg trace formula and related topics (Brunswick, Maine, 1984), Contemp. Math., vol. 53, Amer. Math. Soc., Providence, RI, 1986, pp. 141–163.

[11] Ramesh Gangolli, *The length spectra of some compact manifolds of negative curvature*, J. Differential Geom. **12** (1977), no. 3, 403–424.

[12] Dennis A. Hejhal, *The Selberg trace formula for* $\mathrm{PSL}(2, \mathbf{R})$. *Vol. 2*, Lecture Notes in Mathematics, vol. 1001, Springer-Verlag, Berlin, 1983.

[13] Werner Hoffmann, *An invariant trace formula for the universal covering group of* $\mathrm{SL}(2, \mathbf{R})$, Ann. Global Anal. Geom. **12** (1994), no. 1, 19–63.

[14] Teruaki Kitano, *Reidemeister torsion of Seifert fibered spaces for* $\mathrm{SL}(2; \mathbf{C})$*-representations*, Tokyo J. Math. **17** (1994), no. 1, 59–75.

[15] Alexander S. Markus, *Introduction to the spectral theory of polynomial operator pencils*, Translations of Mathematical Monographs, vol. 71, American Mathematical Society, Providence, RI, 1988.

[16] Henri Moscovici and Robert J. Stanton, *R-torsion and zeta functions for locally symmetric manifolds*, Invent. Math. **105** (1991), no. 1, 185–216.

[17] Werner Müller, *A Selberg trace formula for non-unitary twists*, Int. Math. Res. Not. IMRN **2011** (2011), no. 9, 2068–2109.

[18] ———, *The asymptotics of the Ray-Singer analytic torsion of hyperbolic 3-manifolds*, Metric and differential geometry, Progr. Math., vol. 297, Birkhäuser/Springer, Basel, 2012, pp. 317–352.

[19] Werner Müller, *Ruelle zeta functions of hyperbolic manifolds and Reidemeister torsion*, The Journal of Geometric Analysis (2021), 1–24.

[20] Werner Müller and Jonathan Pfaff, *Analytic torsion of complete hyperbolic manifolds of finite volume*, J. Funct. Anal. **263** (2012), no. 9, 2615–2675.

[21] Liviu I. Nicolaescu, *The Reidemeister torsion of 3-manifolds*, De Gruyter studies in mathematics 30, Walter de Gruyter, 2003.

[22] Shu Shen, *Analytic torsion, dynamical zeta functions, and the Fried conjecture*, Anal. PDE **11** (2018), no. 1, 1–74.

[23] ______, *Complex valued analytic torsion and dynamical zeta function on locally symmetric spaces*, (2020), preprint, available at arXiv:2009.03427.

[24] ______, *Analytic Torsion, Dynamical Zeta Function, and the Fried Conjecture for Admissible Twists*, Commun. Math. Phys. **387** (2021), no. 2, 1215–1255.

[25] Polyxeni Spilioti, *Twisted Ruelle zeta function and complex-valued analytic torsion*, (2020), preprint, available at arXiv:2004.13474.

[26] Vladimir G. Turaev, *Euler structures, nonsingular vector fields, and torsions of Reidemeister type*, Math. USSR, Izv. **34** (1990), no. 3, 627–662 (English).

[27] ______, *Introduction to combinatorial torsions. Notes taken by Felix Schlenk*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2001.

[28] Nolan R. Wallach, *On the Selberg trace formula in the case of compact quotient*, Bull. Amer. Math. Soc. **82** (1976), no. 2, 171–195.

[29] Artur Wotzke, *Die Ruellesche Zetafunktion und die analytische Torsion hyperbolischer Mannigfaltigkeiten*, PhD thesis, Bonn, Bonner Mathematische Schriften, 2008, available at https://bib.math.uni-bonn.de/downloads/bms/BMS-389.pdf.

[30] Yoshikazu Yamaguchi, *Dynamical zeta functions for geodesic flows and the higher-dimensional Reidemeister torsion for Fuchsian groups*, (2020), preprint, available at arXiv:2002.01156.

L. Bénard: Mathematisches Institut, Georg–August Universität Göttingen, Bunsenstraße 3–5, 37073 Göttingen, Germany
E-mail address: leo.benard@mathematik.uni-goettingen.de

J. Frahm: Department of Mathematics, Aarhus University, Ny Munkegade 118, 8000, Aarhus C, Denmark
E-mail address: frahm@math.au.dk

P. Spilioti: Department of Mathematics, Aarhus University, Ny Munkegade 118, 8000, Aarhus C, Denmark
E-mail address: spilioti@math.au.dk